PROVING THE REGULARITY OF THE REDUCED BOUNDARY OF PERIMETER MINIMIZING SETS WITH THE DE GIORGI LEMMA

Presented by Antonio Farah

In partial fulfillment of the requirements for graduation with the Dean's Scholars Honors Degree in Mathematics Honors, Department of Mathematics

Dr. Stefania Patrizi, Supervising Professor

Dr. David Rusin, Honors Advisor, Department of Mathematics

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2020

Acknowledgements

I deeply appreciate the continued guidance and support of Dr. Stefania Patrizi, who supervised the research and preparation of this Honors Thesis. Dr. Patrizi kindly dedicated numerous sessions to this project, motivating and expanding my learning.

I also greatly appreciate the support of Dr. Irene Gamba and of Dr. Francesco Maggi on the project, who generously helped with their reading and support. Further, I am grateful to Dr. David Rusin, who also provided valuable help on this project in his role of Honors Advisor in the Department of Mathematics.

Moreover, I would like to express my gratitude to the Department of Mathematics' Faculty and Administration, to the College of Natural Sciences' Administration, and to the Dean's Scholars Honors Program of The University of Texas at Austin for my educational experience.

Abstract

The Plateau problem consists of finding the set that minimizes its perimeter among all sets of a certain volume. Such set is known as a minimal set, or perimeter minimizing set. The problem was considered intractable until the 1960's, when the development of geometric measure theory by researchers such as Fleming, Federer, and De Giorgi provided the necessary tools to find minimal sets. After the existence of minimal sets was proven, the study of perimeter minimizing sets became an active area of mathematical research–focused on determining the properties of minimal sets. One of the most prominent research problems sought to determine how smooth the boundary of minimal sets is in n dimensions, which is also known as the set's regularity. This paper approaches the study of perimeter minimizing sets using geometric measure theory, concluding on the De Giorgi Lemma–which demonstrates that minimal sets have some level of regularity.

Background

The problem of minimal sets consists of finding a surface of least area among all the surfaces with a fixed boundary. J.L. Lagrange first formulated the problem in 1760. In 1849, J. Plateau showed that a soap film stretched on a wire framework is actually a minimal surface; thus, the problem became known as the Plateau problem. In the following years, some mathematicians obtained solutions for specific boundaries, but it was only until 1930 that J. Douglas and T. Rado independently achieved general solutions in $\mathbb{R}^3$. The extension of the problem to higher dimensions was more difficult to study. It took thirty more years to attack the problem in its full generality by using measure-theoretic methods [1].

The first fundamental result for surfaces in codimension 1 was obtained by E. De Giorgi, who showed that there exist solutions to Plateau's problem that are almost everywhere real analytic [6]. In De Giorgi's formalism, a surface in $\mathbb{R}^n$ is seen as the boundary of a measurable set E whose characteristic function φ_E has distributional derivatives that are Radon measures of finite total variation (a set with these properties is known as a Caccioppoli set). The $n-1$ dimensional area of the set is then defined to be the total variation of $D\varphi_E$, also known as the perimeter of E. While it is not difficult to prove that Plateau's problem can be solved in a weak sense, proving that the surfaces obtained are actually regular up to a closed singular set is more difficult. The key step towards proving a regularity result for minimal sets is the De Giorgi Lemma.

The lemma's main idea is to define for any $x \in \partial E$ an approximate normal vector v_ρ as follows: $v_\rho = \frac{\int_{B(0,\rho)} D\varphi_E}{\int_{B(0,\rho)} |D\varphi_E|}$. The lemma states that if E is a minimal set and for some $x \in \partial E, v_\rho$ length close to 1 for some $\rho > 0$, then in fact v_ρ tends to 1 as ρ approaches 0. And if a point x satisfies the assumptions of the lemma, then such point is said to be in the reduced boundary of the minimal set E. The lemma's main consequence is that the reduced boundary of a minimal set is open and locally analytic. Its complement is a closed set of H_{n-1} measure 0.

Our work sets to demonstrate the existence of perimeter minimizing sets and to determine their regularity properties by following the definitions, theorems, and proofs of the work of Enrico Giusti, which presents the notions of De Giorgi on the subject [1]. First, we prove the existence of minimal sets using the properties of a special class of functions known as functions of bounded variation. Second, we obtain a regularity result for perimeter minimizing sets through several tasks. We split the boundary of minimal sets into two parts: the reduced boundary and its complement. We demonstrate the regularity of the reduced boundary using tools from measure theory. We then prove the De Giorgi Lemma, which is a fundamental lemma that specifies the decay rate of the perimeter of a set localized around a point. Once the De Giorgi Lemma is proven, we show that the reduced boundary of a perimeter minimizing sets is locally analytic.

The following are the main results of De Giorgi's work [2] [3] [4] [5] [6], as compiled by Giusti [1].

Table of Contents

Sections

1 Functions of Bounded Variation and Caccioppoli Sets

This section introduces a class of functions that permeate the study of perimeter minimizing sets and geometric measure theory: the functions of bounded variation (denoted from here on as BV functions). Such class of functions is at once both general enough to include most functions of interest in perimeter minimizing sets and narrow enough so that functions of bounded variation enjoy nice properties.

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be open. If $f \in L^1(\Omega)$, we define

$$\int_\Omega |Df| = \sup\{\int_\Omega f(\boldsymbol{\nabla} \cdot g)dx\} \tag{1.1}$$

where the supremum is taken over all $g \in C_0^1(\Omega;\mathbb{R}^n)$ with $|g(x)| \leq 1$ for $x \in \Omega$, and $\boldsymbol{\nabla} \cdot g = \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}$.

Remark 1.1. The purpose of this definition is to provide a nice generalization of what the quantity $\int_\Omega |\nabla f| dx$ means for functions that are not necessarily continuous. The following example demonstrates that if $f \in C^1(\Omega)$, then the two quantities indeed coincide.

Example 1.1. Suppose $f \in C^2(\Omega)$. Then, we have that

$$\int_\Omega f\boldsymbol{\nabla} \cdot g dx = -\int_\Omega \sum_{i=1}^n \frac{\partial f}{\partial x_i} g_i dx = -(\nabla f, g) \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Now, by the Cauchy-Schwarz Inequality, we have that $\int_\Omega \nabla f \cdot g dx \leq \int_\Omega |\nabla f||g| dx \leq \int_\Omega |\nabla f| dx$.

The Cauchy-Schwarz Inequality will only yield equality if g is a scalar multiple of ∇f, and so we would like to set $g = \frac{\nabla f}{|\nabla f|}$ to achieve equality. However, g defined in such way will not have compact support, and so we cannot use this definition of g. In order to resolve this difficulty, let $\{\Omega_n\}_{n=1}^\infty$ be a sequence of increasing sets such that $\Omega_n \subset \Omega$ for all n and $\bigcup_{n=1}^\infty \Omega_n = \Omega$, and let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of small numbers that converge to 0. Also, let t_n be a smooth cut off function that is 0 outside of Ω_n and is 1 on the set $\Omega_{n,\epsilon_n} := \{x \in \Omega_n : \text{dist}(x, \partial\Omega_n) > \epsilon_n\}$. Then, we define $g_n = \frac{\nabla f}{|\nabla f|} t_n$. Next, we use the Monotone Convergence Theorem to get that $\lim_{n\to\infty}(\nabla f, g_n) = \int_\Omega |\nabla f| dx$. Hence, $\int_\Omega |Df| \le \int_\Omega |\nabla f| dx$.

Now, since $C^2(\Omega)$ is dense in $C^1(\Omega)$, and Ω is a bounded domain, it follows that for any $h \in C^1(\Omega)$ there exists a sequence of $C^2(\Omega)$ functions $\{h_j\}_{j=1}^{\infty}$ such that $h_j \to h$ in the $C^1(\Omega)$ norm. Thus, the result holds true for $C^1(\Omega)$, as claimed.

Remark 1.2. The previous fact also holds if $f \in W^{1,1}(\Omega)$, except that now $\frac{\partial f}{\partial x_i}$ are the distributional derivatives of f.

Definition 1.2. If $f \in L^1(\Omega)$ and $\int_\Omega |Df|$ is finite, then we say that f has bounded variation in Ω, and we write $f \in BV(\Omega)$.

Remark 1.3. It must be noted that although BV functions do enjoy nice properties, which we will describe later, BV functions can behave badly in other ways. The following example sheds light on just how general the class of BV functions is.

Example 1.2. The first example showed that $W^{1,1}(\Omega) \subset BV(\Omega)$. We now show that the inclusion is strict. Let $E \subset \mathbb{R}^n$ be bounded with C^2 boundary. We let $\varphi_E(x)$ denote the characteristic function of E. The distributional derivative of φ_E, denoted by $D\varphi_E$, is not a regular distribution. As a result, $\varphi_E(x)$ does not belong to the Sobolev space $W^{1,1}(\Omega)$.

However, $\varphi_E(x)$ is a BV function. To see this, let $g \in C_0^1(\Omega)$. Through the use of the Green-Gauss Theorem, we have that

$$\int_\Omega \varphi_E \boldsymbol{\nabla} \cdot g dx = \int_E \boldsymbol{\nabla} \cdot g dx = \int_{\partial E} g \cdot v dH_{n-1} \tag{1.3}$$

where v is the inner normal vector to ∂E, and H_{n-1} is the $n-1$ Hausdorff measure. Then, we get that

$$\int_{\partial E} g \cdot v dH_{n-1} \le \int_{\partial E} |g||v| dH_{n-1} \le \int_{\partial E} \varphi_\Omega dH_{n-1} = H_{n-1}(\Omega \cap \partial E) < \infty \tag{1.4}$$

and so, $\varphi_E(x) \in BV(\Omega)$.

Remark 1.4. Actually, it is the case that

$$\int_\Omega |D\varphi_E| = H_{n-1}(\partial\Omega \cap E). \tag{1.5}$$

Since $\int_\Omega |D\varphi_E| \leq H_{n-1}(\partial\Omega \cap E)$, we only need to show that the reverse inequality also holds. Notice that, since E has a C^2 boundary, the outward pointing normal $v(x)$ is a $C^1(\Omega)$ function that may be extended to the whole of $\mathbb{R}^n$ and satisfies $|v(x)| \leq 1$ for all $x \in \mathbb{R}^n$. Now, if $\eta \in C_0^\infty(\Omega)$ with $|\eta(x)| \leq 1$ for all $x \in \Omega$, then it follows that

$$\int_E \boldsymbol{\nabla} \cdot g dx = \int_{\partial E} \eta dH_{n-1} \tag{1.6}$$

where $g = v\eta$. Hence,

$$\int_\Omega |D\varphi_E| \geq \sup\{\int_{\partial\Omega} \eta dH_{n-1} : \eta \in C_0^\infty(\Omega), |\eta| \leq 1\} = H_{n-1}(\partial E \cap \Omega). \tag{1.7}$$

Remark 1.5. If $f \in BV(\Omega)$, then we can define a measure v on any Borel set Ω by setting $v(\Omega) = \int_\Omega |Df|$. Since f is locally integrable and $\sup_g(f\boldsymbol{\nabla} \cdot g)$ for $g \in C_0^\infty(\mathbb{R}^n, \Omega)$ is non-negative, it follows that v is a Radon measure. In this way, we can define what the quantity $\int_\Omega |Df|$ means when Ω is not necessarily open. Specifically, if Ω is not open but rather Borel, we set $\int_\Omega |Df| = \lim_{n\to\infty} \int_{\Omega_n} |Df|$, where $\{\Omega_n\}_{n=1}^\infty$ is a sequence of decreasing sets with $\Omega \subset \Omega_n$ for all n such that $\mathcal{L}^n(\Omega_n - \Omega) \to 0$ as $n \to \infty$. For such reason, we will typically work with open sets, and the same properties hold true for Borel sets by passing to the limit.

Now we have the necessary tools to rigorously introduce the notion of a perimeter.

Definition 1.3. Let E be a Borel set and let Ω be an open set in $\mathbb{R}^n$. We define the perimeter of E in Ω, denoted by $P(E, \Omega)$, as follows:

$$P(E, \Omega) = \int_\Omega |D\varphi_E|. \tag{1.8}$$

If $\Omega = \mathbb{R}^n$, we simply write $P(E, \Omega) = P(E)$.

The following definition introduces a class of sets that is crucial to the study of minimal sets.

Definition 1.4. If $P(E, \Omega)$ is finite for every bounded open set Ω, then E is said to be a Caccioppoli set.

Remark 1.6. if E, E_1, and E_2 are Caccioppoli sets, then it follows that

$$\Omega \subset \Omega_1 \to P(E,\Omega) \leq P(E,\Omega_1) \tag{1.9}$$

with equality holding if and only if $E \subset\subset \Omega$. Also, we have that

$$P(E_1 \cup E_2, \Omega) \leq P(E_1,\Omega) + P(E_2,\Omega) \tag{1.10}$$

with equality holding when $\mathrm{dist}(E_1, E_2) > 0$. Finally, if $L^n(E) = 0$, then $P(E) = 0$, and if $\mathcal{L}^n(E_1 \triangle E_2) = 0$, then $P(E_1) = P(E_2)$.

Remark 1.7. If $E \subset \Omega$ is a Caccioppoli set, then the definition of $D\varphi_E$ implies that if we set $\omega = -D\varphi_E$, then

$$\int_E \boldsymbol{\nabla} \cdot g dx = \int g \cdot d\omega \tag{1.11}$$

for any $g \in C_0^1(\Omega, \mathbb{R}^n)$. Hence, $D\varphi_E$ acts as a vector-valued Radon measure with locally finite variation.

We note here that the converse of such statement is also true. If there exists a vector-valued Radon measure with locally finite variation ω such that $\int_E \boldsymbol{\nabla} \cdot g dx = \int g d\omega$, then it follows that if $g \in C_0^1(\Omega, \mathbb{R}^n)$ with $|g(x)| \leq 1$, then

$$\int_E \boldsymbol{\nabla} \cdot g dx = \int g \cdot d\omega \leq |\omega|(\Omega) < \infty \tag{1.12}$$

and so E is a Caccioppoli set.

Remark 1.8. In some sense to be made clear here, $D\varphi_E \neq 0$ only on ∂E. Being more precise, we extend the definition of the support of a function as follows:

$$\mathrm{spt} D\varphi_E = \mathbb{R}^n - \cup\{\text{open sets } A \text{ such that } g \in C_0^1(A; \mathbb{R}^n) \Rightarrow \int g \cdot D\varphi_E = 0\}. \tag{1.13}$$

Now, suppose $x \notin \partial E$. Then, we can find some open set A such that $x \in A$ and $A \cap \partial E = \emptyset$. As a result, if $g \in C_0^1(A, \mathbb{R}^n)$, then $\int g \cdot D\varphi_E = -\int_E \boldsymbol{\nabla} \cdot g dx = 0$.

Combining the two previous remarks, we have derived a Gauss-Green formula for Caccioppoli sets, namely that, if $g \in C_0^1(\Omega, \mathbb{R}^n)$ with $|g| \leq 1$, then

$$\int_E \boldsymbol{\nabla} \cdot g dx = -\int_{\partial E} g \cdot D\varphi_E. \tag{1.14}$$

Next, we present the analogue of Fatou's lemma for functions of bounded variation. Such theorem will be fundamental in the work ahead.

Theorem 1.1. *(Semicontinuity) Let $\Omega \subset \mathbb{R}^n$ be an open set and $\{f_j\}$ a sequence of functions in $BV(\Omega)$ which converge in $L^1_{loc}(\Omega)$ to a function f. Then,*

$$\int_\Omega |Df| \le \lim \inf_{j\to\infty} \int_\Omega |Df_j|. \tag{1.15}$$

Proof. Let $g \in C^1_0(\Omega, \mathbb{R}^n)$ with $|g(x)| \le 1$. Then, it follows that

$$\int_\Omega f \boldsymbol{\nabla} \cdot g dx = \lim_{j\to\infty} \int_\Omega f_j \boldsymbol{\nabla} \cdot g dx \le \lim \inf_{j\to\infty} \int_\Omega |Df_j|. \tag{1.16}$$

Next, by taking the supremum over all such g, we obtain the result. □

Earlier, it was mentioned that if a set E has some level of regularity (in this case C^2), then $P(E,\Omega) = H_{n-1}$. Now, we will investigate an example showing that this is not true in general.

Example 1.3. Let $\{x_j\}_{j=1}^\infty$ be the set of all rational points in $\mathbb{R}^n$. Then, we let $B = \bigcup_{j=0}^\infty B(x_j, 2^{-j})$. Now, we compute that

$$\mathcal{L}^n(B) \le \sum_{j=1}^\infty \mathcal{L}^n(B(x_j, 2^{-j})) = \frac{\gamma_n}{1-2^{-n}},$$

where γ_n is the volume of the unit ball in $\mathbb{R}^n$. Also, since the rationals are dense in $\mathbb{R}^n$, it follows that $\mathcal{L}^n(\partial B) = \infty$, and so $H_{n-1}(\partial B) = \infty$.

Now, let us define $B_k = \bigcup_{j=1}^k B(x_j, 2^{-j})$. Then, it follows that $\varphi_{B_k} \to \varphi_B$ in $L^1(\mathbb{R}^n)$. And since each B_k is smooth, we compute that

$$P(B_k) = H_{n-1}(B_k) \le \sum_{j=0}^n H_{n-1} B_k = n\gamma_{n-1} \sum_{j=0}^k 2^{-j(n-1)} \le n\gamma_{n-1} \sum_{j=0}^\infty 2^{-j(n-1)} \tag{1.17}$$

$$= \frac{n\gamma_{n-1}}{1-2^{-(n-1)}}. \tag{1.18}$$

And so we have that $P(B) \le \lim\inf_{j\to\infty} P(B_k) = \frac{n\gamma_{n-1}}{1-2^{-(n-1)}} < \infty$.

Remark 1.9. If we define

$$||f||_{BV} = ||f||_{L^1} + \int_\Omega |Df|, \tag{1.19}$$

then it is indeed true that $BV(\Omega)$ is a Banach space.

Now, we present somewhat of an analogue to the Semicontinuity theorem, but dealing with the lim sup rather than the lim inf.

Theorem 1.2. *Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of $BV(\Omega)$ functions such that $f_j \to f$ in $L^1_{loc}(\Omega)$ and that*

$$\lim_{j\to\infty} \int_\Omega |Df_j| = \int_\Omega |Df|. \tag{1.20}$$

Then, if $A \subset \Omega$ is open,

$$\int_{\bar{A}\cap\Omega} |Df| \geq \limsup_{j\to\infty} \int_{\bar{A}\cap\Omega} |Df_j|. \tag{1.21}$$

And if

$$\int_{\partial A\cap\Omega} |Df| = 0, \tag{1.22}$$

then

$$\int_A |Df| = \lim_{j\to\infty} \int_A |Df_j|. \tag{1.23}$$

Proof. Let $B = \Omega - \bar{A}$. Since $\bar{A}$ is closed, B is open, and so semicontinuity implies that

$$\int_A |Df| \leq \liminf_{j\to\infty} \int_A |Df_j| \tag{1.24}$$

$$\int_B |Df| \leq \liminf_{j\to\infty} \int_B |Df_j|. \tag{1.25}$$

Also, we compute that

$$\int_{\bar{A}\cap\Omega} |Df| + \int_B |Df| = \int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j| \tag{1.26}$$

$$\geq \limsup_{j\to\infty} \int_{\bar{A}\cap\Omega} |Df_j| + \liminf_{j\to\infty} \int_B |Df_j| \tag{1.27}$$

$$\geq \limsup_{j\to\infty} \int_{\bar{A}\cap\Omega} |Df_j| + \int_B |Df|. \tag{1.28}$$

Consequently, it follows that $\int_{\bar{A}\cap\Omega} |Df| \geq \limsup_{j\to\infty} \int_{\bar{A}\cap\Omega} |Df_j|$.

Next, we assume that $\int_{\partial A\cap\Omega} |Df| = 0$. Then, since $A - \partial A$ is compactly contained in Ω, and since $\int_\Omega |Df| = \int_{A-\partial A} |Df| = \lim_{j\to\infty} \int_{A-\partial A} |Df_j| + \int_{\partial A} |Df_j|$, it follows that $\lim_{j\to\infty} \int_{\partial\Omega} |Df_j| = 0$, and so $\int_A |Df| = \lim_{j\to\infty} \int_A |Df_j|$. □

Now, we would like to find a way to approximate BV functions by functions that have better properties. For this purpose, we introduce mollifier functions. A function $\eta : \mathbb{R}^n \to \mathbb{R}$ is a mollifier if and only if it satisfies the following properties:

1. $\eta(x) \in C_0^\infty(\mathbb{R}^n)$
2. η is compactly supported in $B_1(0)$
3. $\int_{\mathbb{R}^n} \eta(x)dx = 1$

Also, η is called a positive symmetric mollifier if it additionally satisfies the following properties:

1. $\eta(x) \geq 0$.
2. $\eta(x) = \mu(|x|)$ for some function $\mu : \mathbb{R}^+ \to \mathbb{R}$.

Now, if $f \in L^1_{loc}(\mathbb{R}^n)$, η is a positive symmetric mollifier, and $\epsilon > 0$, we define:

$$\eta_\epsilon(x) = \epsilon^{-n}\eta(\frac{x}{\epsilon}) \tag{1.29}$$

$$f_\epsilon = \eta_\epsilon \circledast f. \tag{1.30}$$

By definition, f_ϵ will have the following properties:

1. $f_\epsilon \in C^\infty(\mathbb{R}^n)$, $f_\epsilon \to f$ in $L^1_{loc}(\mathbb{R}^n)$.
2. If $f \in L^1(\mathbb{R}^n)$, then $f_\epsilon \to f$ in $L^1(\mathbb{R}^n)$.
3. If there exist $A, B \in \mathbb{R}$ such that $A \leq f(x) \leq B$ for all $x \in \mathbb{R}^n$, then $A \leq f_\epsilon(x) \leq B$ for all $x \in \mathbb{R}^n$.

4. If $f, g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f_\epsilon g dx = \int_{\mathbb{R}^n} f g_\epsilon dx$.
5. If $f \in C^1(\mathbb{R}^n)$, then $\frac{\partial f_\epsilon}{\partial x_i} = (\frac{\partial f}{\partial x_i})_\epsilon$.
6. If $\mathrm{spt} f \subset A$, then $\mathrm{spt} f_\epsilon \subset A_\epsilon = \{x : \mathrm{dist}(x, A) \leq \epsilon\}$.

The next theorem gives precise conditions that allow us to approximate the BV-seminorm of a BV function by the BV seminorms of the mollifications of such functions.

Theorem 1.3. *Let $f \in BV(\Omega)$, and suppose that $A \subset\subset \Omega$ is an open set such that*

$$\int_{\partial A} |Df| = 0. \tag{1.31}$$

Then, we have that

$$\int_A |Df| = \lim_{\epsilon \to 0} \int_A |Df_\epsilon|. \tag{1.32}$$

Proof. By the semicontinuity theorem, we have that

$$\int_A |Df| \leq \liminf_{\epsilon \to 0} \int_A |Df_\epsilon|. \tag{1.33}$$

Now, choose some $g \in C_0^1(A, \mathbb{R}^n)$ with $|g(x)| \leq 1$. Then, the properties of mollifiers imply that

$$\int_\Omega f_\epsilon \boldsymbol{\nabla} \cdot g dx = \int_\Omega f \boldsymbol{\nabla} \cdot g_\epsilon dx = \int_{A_\epsilon} f \boldsymbol{\nabla} \cdot g_\epsilon dx. \tag{1.34}$$

Hence, it follows that

$$\int_A f_\epsilon \boldsymbol{\nabla} \cdot g dx \leq \int_{A_\epsilon} |Df|, \tag{1.35}$$

and so,

$$\int_A |Df_\epsilon| \leq \int_{A_\epsilon} |Df|. \tag{1.36}$$

Thus, it follows that

$$\limsup_{\epsilon \to 0} \int_A |Df_\epsilon| \leq \lim_{\epsilon \to 0} \int_{A_\epsilon} |Df| = \int_{\bar{A}} |Df|. \tag{1.37}$$

Then, our assumption that $\int_{\partial A} |Df| = 0$ implies that

$$\limsup_{\epsilon \to 0} \int_A |Df_\epsilon| \leq \int_A |Df|. \tag{1.38}$$

Consequently,

$$\int_A |Df| = \lim_{\epsilon \to 0} \int_A |Df_\epsilon|. \tag{1.39}$$

□

The following theorem provides a way to approximate BV functions by C^∞ functions. Since the closure of $C^\infty(\Omega)$ functions in the BV norm is the Sobolev space $W^{1,1}(\Omega)$, we cannot approximate BV functions by C^∞ functions in this norm. Nevertheless, L^1 convergence and convergence of the BV-seminorms is indeed possible.

Theorem 1.4. *Suppose that $f \in BV(\Omega)$. Then, there exists a sequence of C^∞ functions $\{f_j\}_{j=1}^\infty$ such that $f_j \to f$ in $L^1(\Omega)$ and*

$$\lim_{j \to \infty} \int_\Omega |Df_j| = \int_\Omega |Df|. \tag{1.40}$$

Actually, it is possible to prove even a stronger result: if $x \in \Omega$, then

$$\lim_{p \to 0} p^{-n} \int_{B_p(x) \cap \Omega} |f_j - f| dx = 0 \tag{1.41}$$

where f_j is any of the sequence of C^∞ functions that approximate f in the previous theorem. Now, we present a compactness result. Such result will be essential in the next theorem, which will prove the existence of perimeter minimizing sets, i.e., Caccioppoli sets that minimize perimeter in a given bounded domain.

Theorem 1.5. *(Compactness) Let Ω be a bounded open set in $\mathbb{R}^n$ with at least a Lipschitz boundary. Then, if $\{f_\alpha\}_{\alpha \in I}$ is a family of functions such that $||f_\alpha||_{BV} \leq M$ for some $M \in \mathbb{R}$ for all $\alpha \in I$, then such family is relatively compact in $L^1(\Omega)$ (that is, every sequence admits a convergent subsequence in $L^1(\Omega)$).*

Proof. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of the uniformly bounded family. Then, by the preceding theorem, for any $\epsilon > 0$ we can find some sequence of C^{∞} functions $\{g_j\}_{j=1}^{\infty}$ such that

$$\int_{\Omega} |f_j - g_j| \leq \frac{1}{j} \qquad \int_{\Omega} |Dg_j| \leq M + \epsilon. \tag{1.42}$$

Then, the Rellich-Kondrachonov theorem implies that $\{g_j\}_{j=1}^{\infty}$ is compactly embedded in $L^1(\Omega)$. Hence, there is some subsequence $\{g_{m(j)}\}_{j=1}^{\infty}$ converging to some $f \in L^1(\Omega)$. We easily compute that

$$\int_{\Omega} |f_{m(j)} - f|dx \leq \int_{\Omega} |f_{m(j)} - g_{m(j)}|dx + \int_{\Omega} |g_{m(j)} - f|dx \to 0. \tag{1.43}$$

Thus, $\{f_{m(j)}\}_{j=1}^{\infty} \to f$. Also, $\int_{\Omega} |Df| \leq \liminf_{j\to\infty} \int_{\Omega} |Df_{m(j)}| \leq M < \infty$, and so $f \in BV(\Omega)$. Hence, $\{f_{\alpha}\}_{\alpha\in I}$ is relatively compact in $L^1(\Omega)$. □

Now, we present the culmination of the work shown so far: the existence of minimal sets!

Theorem 1.6. *(Existence of minimal sets) Let Ω be a bounded open set in $\mathbb{R}^n$, and let L be a Caccioppoli set. Then, there exists a set E coinciding with L outside Ω and such that*

$$\int |D\varphi_E| \leq \int |D\varphi_F|$$

for every set F with $F = L$ outside Ω.

Remark 1.10. Before presenting the proof of such theorem, it is important to highlight that the proof presents an approach that is prevalent throughout the study of perimeter minimizing sets. The steps of the approach will be highlighted clearly in the proof. Also, we can think of this problem as a partial differential equation. The minimality property can be seen as the property that a function solving a partial differential equation would need to have to be a solution, while L serves to enforce some sort of a boundary condition.

Proof. Before presenting the approach, we note that since Ω is bounded, it is compactly contained in some ball $B_R(0)$. Hence, since

$$\int |D\varphi_F| = \int_{B_R(0)} |D\varphi_F| + \int_{B_R(0)^c} |D\varphi_F| \tag{1.44}$$

for any set F, and we are only concerned with sets that coincide with L outside Ω, it follows that we only need to find a set E coinciding with L outside Ω such that

$$\int_{B_R(0)} |D\varphi_E| \leq \int_{B_R(0)} |D\varphi_F| \tag{1.45}$$

for all sets F coinciding with L outside Ω.

Let A be the set of all sets coinciding with L outside Ω. The first step of the proof is to take a sequence $\{E_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \int_{B_R(0)} |D\varphi_{E_j}| = k$, where $k = \inf_{F\in A} \int_{B_R(0)} |D\varphi_F|$, and show that the sequence is uniformly bounded in $BV(\Omega)$. Since the sequence is minimizing, we have that $\int_{B_R(0)} |\varphi_{E_j}|$ is uniformly bounded. Also, since $B_R(0)$ is bounded, $\int_{B_R(0)} |D\varphi_{E_j}|$ is uniformly bounded as well. Hence, the minimizing sequence is uniformly bounded.

The second step is to use our compactness theorem to show that $\varphi_{E_j} \to f$, for some subsequence still denoted by E_j and for some $L^1(\Omega)$ function f. Since φ_{E_j} is 0 or 1 everywhere, we have that, by taking another subsequence if neccesary, f is also the characteristic function of some set E coinciding with L outside Ω (note that a convergent sequence in $L^1(\Omega)$ admits a subsequence converging pointwise a.e.).

The third and final step is to use our semicontinuity theorem to show that E enjoys the minimality property.

Thus, we have proven the existence of minimal sets. $\square$

We showed earlier that BV functions can be approximated by C^∞ functions in some way. To conclude this section, we will prove that a similar result holds for Caccioppoli sets and C^∞ sets. Before that, we introduce two auxiliary lemmas.

Lemma 1.1. *(Coarea formula) Suppose that $f \in BV(\Omega)$. Define*

$$F_t = \{x \in \Omega : f(x) < t\}. \tag{1.46}$$

Then, it follows that

$$\int_\Omega |Df| = \int_{-\infty}^{\infty} dt \int_\Omega |D\varphi_{F_t}|. \tag{1.47}$$

Lemma 1.2. *If $t \in (0,1)$, $\epsilon_j \to 0$, and we define*

$$E_j = \{x \in \mathbb{R}^n : f_{e_j} < t\} \tag{1.48}$$

where $f_{\epsilon_j} = \eta_{\epsilon_j} \circledast \varphi_E$, E is a Caccioppoli set and η is a mollifier. Then, it follows that

$$\lim_{j\to\infty} \int |\varphi_{E_j} - \varphi_E| dx \leq \frac{1}{\min(t, 1-t)} \int |f_{\epsilon_j} - \varphi_E| dx. \tag{1.49}$$

Now, we show for Caccioppoli sets and C^∞ sets a result similar to the one shown for approximating BV functions by C^∞ functions in some way.

Theorem 1.7. *Let E be a bounded Caccioppoli set. Then, there exists a sequence of sets $\{E_j\}_{j=1}^\infty$ with a C^∞ boundary such that $\varphi_{E_j} \to \varphi_E$ in $L^1(\mathbb{R}^n)$ and*

$$\int |D\varphi_{E_j}| \to \int |D\varphi_E|. \tag{1.50}$$

Remark 1.11. *The idea of the proof is to use the fact that φ_E can be approximated by a sequence of C^∞ functions. From such functions we will obtain our approximating sets.*

Proof. Set $\epsilon > 0$, and let $f_\epsilon = \varphi_\epsilon \circledast \eta_\epsilon$, where η is a mollifier. Then, since $0 \leq f_\epsilon \leq 1$, the coarea formula implies that

$$\int |Df_\epsilon| = \int_0^1 dt \int |D\varphi_{E_{\epsilon,t}}| \tag{1.51}$$

where $E_{\epsilon,t} = \{x \in \mathbb{R}^n : f_\epsilon(x) < t\}$. Then, Theorem 1.3 implies that

$$\int |D\varphi_E| \leq \lim_{\epsilon\to 0} \int |Df_\epsilon|. \tag{1.52}$$

Now, choose a sequence $\{\epsilon_j\}_{j=1}^\infty$ that tends to 0, and let $t \in (0,1)$. Then, our preceding lemma implies, along with the fact that $f_{\epsilon_j} \to \varphi_E$ in $L^1(\mathbb{R}^n)$, that $\varphi_{E_{\epsilon_j,t}} \to \varphi_E$ in $L^1(\mathbb{R}^n)$. As a result, semicontinuity implies that

$$\liminf_{j\to\infty} \int |D\varphi_{E_{\epsilon_j,t}}| \geq \int |D\varphi_E|. \tag{1.53}$$

Then, we calculate that

$$\int |D\varphi_E| = \lim_{j\to\infty} \int |Df_{\epsilon_j}| \tag{1.54}$$

$$= \lim_{j\to\infty} \int_0^1 dt |D\varphi_{E_{\epsilon_j,t}}| \geq \lim \inf_{j\to\infty} \int_0^1 dt |D\varphi_{E_{\epsilon_j,t}}| \geq \int |D\varphi_E|. \tag{1.55}$$

Hence, it follows that

$$\lim_{j\to\infty} \int |D\varphi_{E_{\epsilon_j,t}}| = \int |D\varphi_E|. \tag{1.56}$$

Finally, Sard's lemma implies that we can choose some $t \in (0,1)$ such that $E_{\epsilon_j,t}$ is C^∞. □

We conclude this section with a lemma controlling the L^1 difference between φ_E and φ_{E_j}

Lemma 1.3. *If $t \in (0,1)$, $\epsilon_j \to 0$ as $j \to \infty$ and*

$$E_j = \{x \in \mathbb{R}^n : f_{\epsilon_j}(x) > t\} \tag{1.57}$$

then we have that

$$\lim_{j\to\infty} \int |\varphi_{E_j} - \varphi_E| dx \leq \frac{1}{\min(t, 1-t)} \int |f_{\epsilon_j} - \varphi_E| dx \tag{1.58}$$

2 Traces

Suppose we have a continuous function on an open set Ω in $\mathbb{R}^n$. Since the function is continuous, we can determine its value on the boundary of the set by setting $f(x_0) = \lim_{x \to x_0} f(x)$ for any $x_0 \in \partial\Omega$. However, if we merely have $f \in L^p(\Omega)$ or $f \in BV(\Omega)$, then it is not such a straightforward task to define an extension from the function to its boundary, since it could be the case that $\mathcal{L}^n(\partial\Omega) = 0$. This section describes the values of a $BV(\Omega)$ function at its boundary, known as the trace of the function.

First, we introduce the Lebesgue Points Theorem, a classical result in analysis having far-reaching implications for this chapter.

Theorem 2.1. *(Lebesgue Points) If $f \in L^1(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{\rho \to 0} \frac{\int_{B(0,\rho)} |f(x+t) - f(x)| dt}{\rho^n} = 0. \tag{2.1}$$

The theorem, while indeed useful, does not allow us to define the trace of a $BV(\Omega)$ function. However, the fact that $BV(\Omega)$ functions are in some sense differentiable allows us to define their trace rigorously, as we explain later. Now, we present a useful covering lemma, known as the Vitali Covering Lemma

Lemma 2.1. *(Vitali Covering) Suppose that $A \in \mathbb{R}^n$ and $\rho : A \to (0, 1)$. Then, there exists some countable collection of points $\{x_j\}_{j=1}^{\infty} \subset A$ such that*

$$i \neq j \to B(x_i, \rho(x_i)) \cap B(x_j, \rho(x_j)) = \emptyset \tag{2.2}$$

$$A \subset \bigcup_{j=1}^{\infty} B(x_j, 3\rho(x_j)). \tag{2.3}$$

Although we generally work in n dimensions, defining traces is analogous to reducing one dimension, since the boundary of an open set in $\mathbb{R}^n$ can be considered to be in $\mathbb{R}^{n-1}$. Hence, we use $B_r(x)$ to refer to an open ball of radius r about x in $\mathbb{R}^n$, and $\mathcal{B}_t(y)$ to refer to a ball of radius t about y in $\mathbb{R}^{n-1}$. Additionally, we define an n-dimensional positive cylinder $C_\rho^+(y)$ as $\mathcal{B}(0,\rho) \times (0,\rho)$, and an n-dimensional negative cylinder $C_\rho^-(y)$ as $\mathcal{B}(0,\rho) \times (-\rho, 0)$. Now, we present a similar theorem to the Lebesgue Points Theorem but for cylinders in $\mathbb{R}^n$.

Lemma 2.2. *Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, and let μ be a positive Radon measure with $\mu(\mathbb{R}^n_+) < \infty$. Then, it follows that for a.e $x \in \mathbb{R}^{n-1}$ with respect to H_{n-1}*

$$\lim_{\rho \to 0} \rho^{1-n} \mu(C^+_\rho(x)) = 0. \tag{2.4}$$

Now, we are ready to rigorously define the trace on cylinders. This is the first step in defining the trace for Caccioppoli sets.

Lemma 2.3. *If C^+_R is a cylinder centered at the origin and $f \in BV(C^+_R)$, then there exists a function $f^+ \in L^1(\mathcal{B}(0,R))$ with the property that for H_{n-1} a.e. y*

$$\lim_{\rho \to 0} \rho^{-n} \int_{C^+_\rho(y)} |f(z) - f^+(y)| dz = 0. \tag{2.5}$$

Also, if $C_R = \mathcal{B}(0,R) \times (-R,R)$, then

$$\int_{C^+_R} f \boldsymbol{\nabla} \cdot g dx = -\int_{C^+_R} \langle Df, g \rangle dx + \int_{\mathcal{B}_R} f^+ g_n dH_{n-1} \tag{2.6}$$

holds for every $g \in C^1_0(C_R, \mathbb{R}^n)$.

Proof. First, assume that $f \in C^\infty(C^+_R)$. We define $f_\epsilon(y) = f(y,\epsilon)$. Then, we let $Q_{\epsilon',\epsilon} = \mathcal{B}(0,R) \times (\epsilon', \epsilon)$. Since $f \in C^\infty(C^+_R)$, we may apply Fubini's theorem to obtain the following:

$$\int_{Q_{\epsilon',\epsilon}} |D_n f| dx = \int_{\mathcal{B}(0,R)} \int_{\epsilon'}^{\epsilon} |D_n f(y,t)| dt dH_{n-1} \tag{2.7}$$

$$\geq \int_{\mathcal{B}(0,R)} |\int_{\epsilon'}^{\epsilon} D_n f(y,t) dt| dH_{n-1} = \int_{\mathcal{B}(0,R)} |f_\epsilon - f_{\epsilon'}| dH_{n-1}. \tag{2.8}$$

Hence, it follows that

$$\int_{\mathcal{B}(0,R)} |f_\epsilon - f_{\epsilon'}| dH_{n-1} \leq \int_{Q_{\epsilon',\epsilon}} |D_n f| dx. \tag{2.9}$$

Then, letting ϵ' and ϵ both tend to 0, it follows that f_ϵ is a Cauchy sequence in $L^1(\mathcal{B}(0,R))$. As a result, the sequence must converge to some function $f^+ \in L^1(\mathcal{B}(0,R))$ as $\epsilon \to 0$.

Now, note that $\partial Q_{\epsilon,R} = \mathcal{B}(0,R) \times \{\epsilon\} \cup \mathcal{B}(0,R) \times \{R\} \cup \partial\mathcal{B}(0,R) \times (\epsilon, \epsilon')$. Hence, if $g \in C^1_0(C_R; \mathbb{R}^n)$, it follows that g vanishes on $\mathcal{B}(0,R) \times \{R\}$ and on

$\partial\mathcal{B}(0,R)\times(\epsilon,\epsilon')$. As a result, the usual integration by parts formula yields that

$$\int_{Q(\epsilon,R)} f\boldsymbol{\nabla}\cdot g dx = \int_{\mathcal{B}(0,R)} f_\epsilon g_{\epsilon n} dH_{n-1} - \int_{Q(\epsilon,R)} Df\cdot g dx, \tag{2.10}$$

where g_n denotes the n-th component of g. Then, by letting $\epsilon\to 0$ and using the dominated convergence theorem, it follows that

$$\int_{C_R^+} f\boldsymbol{\nabla}\cdot g dx = -\int_{C_R^+}\langle Df, g\rangle dx + \int_{\mathcal{B}(0,R)} f^+ g_{\epsilon n} dH_{n-1}. \tag{2.11}$$

Now, we focus on $\int_{C_\rho^+(y)}|f(z)-f^+(y)|dz$. We compute that

$$\int_{C_\rho^+(y)}|f(z)-f^+(y)|dz = \int_{\mathcal{B}(y,\rho)}\int_0^\rho |f(x,t)-f^+(y)|dtdx \tag{2.12}$$

$$\leq \int_{\mathcal{B}(y,\rho)}\int_0^\rho |f(x,t)-f^+(x)|dtdx + \rho\int_{\mathcal{B}(y,\rho)}\int_0^\rho |f^+(x)-f^+(y)|dx. \tag{2.13}$$

Next, notice that the Lebesgue Points Theorem implies that

$$\lim_{\rho\to 0}\rho^{1-n}\int_{\mathcal{B}(y,\rho)}\int_0^\rho |f^+(x)-f^+(y)|dx = 0, \tag{2.14}$$

since $\lim_{\rho\to 0}\rho^{1-n}\int_{\mathcal{B}(y,\rho)}\int_0^\rho |f^+(x)-f^+(y)|dx = 0$. Also, since

$$|Df| = \sqrt{\sum_{i=1}^n |D_i f|^2} \geq D_n f, \tag{2.15}$$

we have that

$$\int_{\mathcal{B}(y,\rho)} |f(x,t)-f^+(x)|dx \leq \int_{C_\rho^+(y)}|Df|. \tag{2.16}$$

Therefore,

$$\int_0^\rho\int_{\mathcal{B}(y,\rho)} |f(x,t)-f^+(x)|dxdt \leq \rho\int_{C_\rho^+(y)}|Df|. \tag{2.17}$$

By the previous lemma, we conclude that $\lim_{\rho\to 0}\rho^{1-n}\int_{C_\rho^+(y)}|Df| = 0$. Hence, $\lim_{\rho\to 0}\rho^{-n}\int_{C_\rho^+(y)}|f(z)-f^+(y)|dz \leq \rho^{1-n}\int_{C_\rho^+(y)}|Df| + \rho^{1-n}\int_{\mathcal{B}(y,\rho)}|f^+(x) - f^+(y)|dt = 0$, and so the theorem is proven when $f \in C^\infty(C_R^+)$.

Now, suppose merely that $f \in BV(C_R^+)$. Then, we have that there exists a sequence of $C^\infty(C_R^+)$ functions $\{f_j\}_{j=1}^\infty$ such that $\int_{C_R^+}|f_j - f|dz \to 0$ and $\int_{C_R^+}|Df_j| \to \int_{C_R^+}|Df|$. By Theorem 1.4 and equation 1.41, we get that

$$\lim_{\rho\to 0}\rho^{-n}\int_{C_\rho^+(y)}|f(z)-f_j^+(z)|dz = 0. \tag{2.18}$$

Additionally, all of the traces of each f_j coincide. Hence, we may define $f^+ = f_j^+$; as a consequence, we get that

$$\lim_{p\to 0}\rho^{-n}\int_{C_\rho^+(y)}|f(z)-f^+(y)|dz = 0. \tag{2.19}$$

Also, notice that since f_j is smooth, it follows that

$$\int_{C_R^+} f_j\nabla\cdot g dx = -\int_{C_R^+} g\cdot\nabla f_j dx + \int_{\mathcal{B}_R} f_j^+ g\cdot n dH_{n-1}. \tag{2.20}$$

Now, recall that the Banach-Alaoglu theorem states that any bounded sequence in a Banach space admits a weakly convergent subsequence. If we define $L_g(f) = \int_{C_R^+} g\cdot\nabla f$, then $L_g \in BV(\Omega)^*$, where $BV(\Omega)^*$ denotes the dual space of $BV(\Omega)$. As a result, by taking a subsequence if necessary, we have that $-\int_{C_R^+} g\cdot\nabla f_j dx \to -\int_{C_R^+} g\cdot\nabla f dx$. Hence, it follows that

$$\int_{C_R^+} f\nabla\cdot g dx = -\int_{C_R^+} g\cdot\nabla f dx + \int_{\mathcal{B}(0,R)} f^+ g\cdot n dH_{n-1}. \tag{2.21}$$

This then concludes the proof. $\square$

The function f^+ is called the trace of f in $\mathcal{B}(0,R)$. Also, notice that since $\lim_{\rho\to 0}\rho^{-n}\int_{C_\rho^+(y)} f(z) - f^+(y)dz = 0$, it follows that $f^+(y) = \lim_{\rho\to 0}\rho^{-n}\frac{\int_{C_\rho^+(y)} f(z)}{\mu(C_\rho^+(y))}$.

Further, note that, in the previous theorem, we proved the result for a BV function by considering an approximating sequence that was infinitely smooth. The next theorem shows that any sequence of BV functions, not just those that are smooth, may be used to define the trace of f.

Theorem 2.2. *Let $f \in BV(C_R^+)$, and let $\{f_j\} \subset BV(C_R^+)$ be a sequence that converges to f in $L^1(C_R^+)$ and such that*

$$\lim_{j\to\infty} \int_{C_R^+} |Df_j| = \int_{C_R^+} |Df|. \tag{2.22}$$

Then, $\lim_{j\to\infty} f_j^+ = f^+$ *and* $f^+ \in L^1(C_R^+)$.

Proof. It was established in the previous theorem that

$$\int_{\mathcal{B}(0,R)} |f(y,\epsilon) - f(y,\epsilon')| dH_{n-1} \le \int_{Q_{\epsilon',\epsilon}} |Df| \tag{2.23}$$

if $\epsilon' < \epsilon$. Hence, letting $\epsilon' < \epsilon$

$$\int_0^{\epsilon'} dt \int_{\mathcal{B}(0,R)} |f^+(y) - f(y,t)| dy \le \int_0^{\epsilon'} dt \int_{Q_{\epsilon',\epsilon}} |Df| = \int_{Q_{\epsilon',\epsilon}} |Df|. \tag{2.24}$$

Now, letting $f_{\epsilon'}(y) = \frac{1}{\epsilon'} \int_0^{\epsilon'} f(y,t)dt$ and $Q_{\epsilon'} = Q_{\epsilon',R}$, it follows, by using Fubini's theorem, that

$$\int_{\mathcal{B}(0,R)} |f^+(y) - f_{\epsilon'}(y)| dy \le \int_{Q_{\epsilon'}} |Df|. \tag{2.25}$$

By the triangle inequality and by the previous computations, it follows that

$$\begin{aligned}\int_{\mathcal{B}(0,R)} |f^+(y) - f_j^+(y)| dy &\le \int_{\mathcal{B}(0,R)} |f^+(y) - f_{\epsilon'}(y)| dy + \int_{\mathcal{B}(0,R)} |f_{\epsilon'}(y) - f_{j,\epsilon'}(y)| dy \\ &\quad + \int_{\mathcal{B}(0,R)} |f_{j,\epsilon'}(y) - f_j^+(y)| dy \qquad (2.26) \\ &\le \int_{\mathcal{B}(0,R)} |Df| dy + \int_{\mathcal{B}(0,R)} |f_{\epsilon'}(y) - f_{j,\epsilon'}(y)| dy + \int_{\mathcal{B}(0,R)} |Df_j| dy. \qquad (2.27)\end{aligned}$$

Now, we compute that

$$\int_{\mathcal{B}(0,R)} |f_{\epsilon'} - f_{j,\epsilon'}| dy \le \frac{1}{\epsilon'} \int_0^{\epsilon'} dt \int_{\mathcal{B}(0,R)} |f(y,t) - f_j(y,t)| dy \to 0 \tag{2.28}$$

as $j \to \infty$. Also, note that Theorem 1.2 implies that $\lim_{j\to\infty} \int_{Q_{\epsilon'}} |Df_j| = \int_{Q_{\epsilon'}} |Df_j|$ for almost every ϵ'. Thus, we have that

$$\limsup_{j\to\infty} \int_{\mathcal{B}(0,R)} |f^+ - f_j^+| dy \le 2 \int_{Q_{\epsilon'}} |Df|. \tag{2.29}$$

Letting $\epsilon' \to R$, we obtain the result. $\square$

Letting $C_R^- = \mathcal{B}(0,R) \cup (-R,0)$, it is possible to define a trace f^- if $f \in BV(C_R^-)$, and f^- satisfies theorems analogous to the previous two. We conclude the section by presenting a theorem highlighting this fact.

Theorem 2.3. *Let $f_1 \in BV(C_R^+)$ and $f_2 \in BV(C_R^-)$. We define $f : C_R \to \mathbb{R}$ by setting $f(x) = f_1(x)$ if $x \in C_R^+$ and $f(x) = f_2(x)$ if $x \in C_R^-$. Then, $f \in BV(C_R)$ and*

$$\int_{\mathcal{B}(0,R)} |f^+ - f^-| dH_{n-1} = \int_{\mathcal{B}(0,R)} |Df|. \tag{2.30}$$

Proof. By Theorem 2.2, and by its analogue for f^-, we have that for $g \in C_0^1(C_R;\mathbb{R})$ with $|g(x)| \leq 1$ for $x \in C_R$,

$$\int_{C_R} f\boldsymbol{\nabla} \cdot g = -\int_{C_R^+} g \cdot Df dx - \int_{C_R^-} g \cdot Df dx + \int_{\mathcal{B}(0,R)} (f^+ - f^-) g \cdot n dH_{n-1}. \tag{2.31}$$

Since all the terms with g on the right-hand side are bounded (recall $|g| \leq 1$) and both f^+ and f^- are in $L^1(\mathcal{B}_R)$, it follows that $f \in BV(C_R)$. Also, due to the compact support of g, we have that

$$\int_{C_R} f\boldsymbol{\nabla} \cdot g dx = -\int_{C_R} \langle Df, g\rangle dx =$$

$$-\int_{C_R^+} \langle Df, g\rangle dx - \int_{C_R^-} \langle Df, g\rangle dx - \int_{\mathcal{B}(0,R)} \langle Df, g\rangle. \tag{2.32}$$

Hence,

$$\int_{\mathcal{B}(0,R)} |f^+ - f^-| g \cdot n dH_{n-1} = \int_{\mathcal{B}_R} f\boldsymbol{\nabla} \cdot g \tag{2.33}$$

from which it follows, by taking the supremum over all $g \in C_0^1(C_R, \mathbb{R})$, that

$$\int_{\mathcal{B}(0,R)} |f^+ - f^-| dH_{n-1} = \int_{\mathcal{B}_R} |Df|. \tag{2.34}$$

□

3 The Reduced Boundary

When dealing with L^p functions, $W^{q,p}$ functions, and BV functions, it is irrelevant to ask about the behavior of the function on sets of measure 0. Indeed, the value of the function on such sets plays no role in determining the function's useful properties, such as its norm. Thus, we can change the value of the function on such sets and still consider it the same function. In this section, we see that this notion extends to Caccioppoli sets; namely, making alterations of measure 0 to the set still yields the same perimeter. Specifically, we separate the boundary of the Caccioppoli set into two parts: the reduced boundary and its complement. The reduced boundary is of crucial importance to prove the regularity of minimal sets. We show that the reduced boundary retains all of the essential properties of Caccioppoli sets.

We begin with the following lemma.

Lemma 3.1. *If E is a Borel set, then there exists a Borel set $\tilde{E}$ such that $\mathcal{L}^n(E\backslash\tilde{E}) = 0$ with the property that*

$$0 < \mathcal{L}^n(\bar{E} \cap B(x,\rho)) < \omega_n \rho^n \tag{3.1}$$

for all $x \in \partial\tilde{E}$, all $\rho > 0$, and where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Note that, by considering $\tilde{E}$ instead of E in the previous proposition, we may assume that $0 < \mathcal{L}^n(\bar{E} \cap B(x,\rho)) < \gamma_n \rho^n$ holds for all $x \in \partial E$ and all $\rho > 0$, since E and $\tilde{E}$ are equivalent in the sense of measure (they differ only by a set of measure 0).

Now, we introduce the reduced boundary of a set E, denoted by $\partial^* E$. This is a particular subset of the boundary of a set, and is of great importance in the study of minimal sets. Indeed, our ultimate goal in this thesis is to show that the reduced boundary is locally analytic.

Definition 3.1. *Suppose that E is a Caccioppoli set. Then, we define $\partial^* E$ as the set of all points $x \in \partial E$ where*

- $\int_{B(x,\rho)} |D\varphi_E| > 0$ *for all $\rho > 0$,*
- $v_\rho(x) := \frac{\int_{B(x,\rho)} D\varphi_E}{\int_{B(x,\rho)} |D\varphi_E|}$ *has a limit $v(x)$ as $\rho \to 0$,*
- $|v(x)| = 1$.

Intuitively, $\partial^* E$ is the "good" part of the boundary, that is, the part of the boundary where there are no singularities. The following example illustrates such fact.

Example 3.1. Suppose that E is a C^1 set; in other words, ∂E is a C^1 hypersurface. Since the boundary has no singularities, we expect $\partial E = \partial^* E$, and this is indeed the case. Equation 1.3 implies that, as measures, $D\varphi_E = v dH_{n-1}$ on ∂E, where v is the inner normal vector. Also, we know that $spt(D\varphi_E) \subset \partial E$, and so it follows that for $x \in \partial E$

$$\int_{B(x,\rho)} D\varphi_E = \int_{B(x,\rho)\cap\partial E} v dH_{n-1}. \tag{3.2}$$

Also, since E is C^1, Remark 1.4 implies that

$$\int_{B(x,\rho)} |D\varphi_E| = H_{n-1}(B(x,\rho) \cap \partial E). \tag{3.3}$$

Consequently, we have that

$$v_\rho(x) = \frac{\int_{B(x,p)\cap\partial E} v dH_{n-1}}{H_{n-1}(B(x,p) \cap \partial E)}. \tag{3.4}$$

Hence, the Lebesgue Points Theorem implies that

$$\lim_{\rho\to 0} v_\rho(x) = v(x), \tag{3.5}$$

and so we have proven that $x \in \partial^* E$. Since x was arbitrary, we conclude that $\partial^* E = \partial E$ if E is a C^1 set.

Example 3.2. Consider the square $E := [0,1] \times [0,1]$ on $\mathbb{R}^2$. We show that $\partial^* E = \partial E \backslash \{(0,0),(0,1),(1,0),(1,1)\}$. By the reasoning of the previous example, we have that the corners of the square are the only points that have a chance of not being in $\partial^* E$. We now prove that $x := (0,0) \notin \partial^* E$; note that the reasoning for the other corners is similar. We have that

$$\int_{B(0,\rho)} D\varphi_E = \int_{B(0,\rho)\cap\{x_2=0\}} (0,1) dH_1 + \int_{B(0,\rho)\cap\{x_1=0\}} (1,0) dH_1 = (\rho,\rho) \tag{3.6}$$

while $\int_{B(0,\rho)} |D\varphi_E| = H_1(B(0,\rho) \cap \partial E) = 2\rho$. Hence, $v_\rho(0) = (\frac{1}{2}, \frac{1}{2})$, with $|v_\rho(0)| = \frac{1}{\sqrt{2}}$. Thus, since $v_\rho(0)$ is independent of ρ, $v_\rho(0) = v(0)$. Then, since $|v(0)| \neq 1$, the third condition of Definition 3.1 is not satisfied, and so $0 \notin \partial^* E$.

Notice that the properties of the reduced boundary are unchanged by translations and rotations. Hence, in the following theorems, we can assume without loss of generality that the origin belongs to the boundary of the set and that the x_1 axis is perpendicular to the set at the origin.

In what follows, let E be a Caccioppoli set. Now, we present a lemma asserting that E and $\mathbb{R}^n \backslash E$ have positive density in $B(0, \rho)$.

Lemma 3.2. *Suppose there exists a number $\bar{\rho}$ such that for every $0 < \rho < \bar{\rho}$*

$$\int_{B(0,\rho)} |D\varphi_E| > 0, \tag{3.7}$$

$$|v_\rho(0)| \geq q > 0. \tag{3.8}$$

Then, if $0 < \rho < \bar{\rho}$, the following estimates hold for some constants C_1, C_2, C_3, and C_4 which depend only on n and q

$$\rho^{-n} \mathcal{L}^n (E \cap B_\rho) \geq C_1 > 0, \tag{3.9}$$

$$\rho^{-n} \mathcal{L}^n ((\mathbb{R}^n \backslash E) \cap B_\rho) \geq C_2 > 0, \tag{3.10}$$

$$0 < C_3 \leq \rho^{1-n} \int_{B(0,\rho)} |D\varphi_E| \leq C_4. \tag{3.11}$$

Before presenting the main result of this section, we introduce an important geometric concept that will be crucial to our work ahead.

Definition 3.2. *If $z \in \partial^* E$, we define the tangent hyperplane T by setting*

$$T(z) = \{x \in \mathbb{R}^n : \langle v(z), x - z \rangle = 0\}, \tag{3.12}$$

as well as its corresponding sub and super level sets, which are

$$T^-(z) = \{x \in \mathbb{R}^n : \langle v(z), x - z \rangle < 0\}, \tag{3.13}$$

$$T^+(z) = \{x \in \mathbb{R}^n : \langle v(z), x - z \rangle > 0\}. \tag{3.14}$$

Theorem 3.1. *Define*

$$E_t = \{x \in \mathbb{R}^n : tx \in E\}. \tag{3.15}$$

Then, as $t \to 0^+$, E_t converges to $T^+(0)$. Also, if A is a set such that $H_{n-1}(\partial A \cap T(0)) = 0$, then

$$\lim_{t \to 0} \int_A |D\varphi_{E_t}| = \int_A |D\varphi_{T^+(0)}| = H_{n-1}(A \cap T(0)). \tag{3.16}$$

Proof. Since the properties under interest are invariant under rotations and translations, we can assume that the inner normal at 0, $v(0)$, is such that $v_1(0) = -1$, and $v_j(0) = 0$ for $j \neq 1$. Then, by definition of the dot product, we have that $T^+(0) = \{x : x_1 < 0\}$.

Let $\rho > 0$. Then, by the change of variables $x \to tx$, we get that

$$\int_{B(0,\rho)} D\varphi_{E_t} = t^{1-n} \int_{B(0,t\rho)} D\varphi_E, \tag{3.17}$$

$$\int_{B(0,\rho)} |D\varphi_{E_t}| = t^{1-n} \int_{B(0,t\rho)} |D\varphi_E|. \tag{3.18}$$

Then, the definition of $v(0)$ implies that

$$\lim_{t\to 0} \frac{\int_{B(0,\rho)} D_1\varphi_{E_t}}{\int_{B(0,\rho)} D_1\varphi_{E_t}} = -1, \tag{3.19}$$

$$\lim_{t\to 0} \frac{\int_{B(0,\rho)} D_i\varphi_{E_t}}{\int_{B(0,\rho)} D_i\varphi_{E_t}} = 0, \tag{3.20}$$

when $i \neq 1$. Also, notice that Lemma 3.1 implies that for some constants C_1 and C_2, $C_1 \leq t^{1-n} \int_{B(0,t\rho)} |D\varphi_E| = \int_{B_\rho} |D\varphi_{E_t}| \leq C_2$, and so we can conclude that $\limsup_{t\to 0} \int_{B(0,\rho)} |D\varphi_{E_t}| < \infty$. Hence, the compactness theorem implies that if $\{t_j\}_{j=1}^{\infty}$ is a sequence converging to 0, then there exists a subsequence, still denoted by $\{t_j\}_{j=1}^{\infty}$ and a Caccioppoli set C such that $\varphi_{E_{t_j}} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$. Furthermore, the De La Vallee Poussin Theorem implies that for almost all ρ,

$$\lim_{j\to\infty} \int_{B_\rho} D\varphi_{E_{t_j}} = \lim_{j\to\infty} \int_{B_p} D\varphi_C. \tag{3.21}$$

By Lemma 3.1, it follows that for almost all ρ,

$$\lim_{j\to\infty} \int_{B_\rho} |D\varphi_{E_{t_j}}| = -\lim_{j\to\infty} \int_{B_\rho} D_1\varphi_{E_{t_j}} = -\int_{B_\rho} D_1\varphi_C. \tag{3.22}$$

Then, the semicontinuity theorem implies that

$$\int_{B_\rho} |D\varphi_C| \leq \lim j \to \infty \int_{B_\rho} |D\varphi_{E_{t_J}} = -\int_{B_\rho} D_1\varphi_C. \tag{3.23}$$

And by the definition of $|D\varphi_C|$, equality holds in the previous estimate. Then, in the sense of measures, it follows that

$$|D\varphi_C| = -D_1\varphi_C \tag{3.24}$$

and that

$$D_i\varphi_C = 0 \tag{3.25}$$

for $i \neq 1$. Consequently, whether $\varphi_C(x) = 1$, where $x = (x_1, ..., x_n)$, only depends on x_1. Also, since φ_C is a non-increasing function of x_1, we can conclude that

$$C = \{x \in \mathbb{R}^n : x_1 < \lambda\} \tag{3.26}$$

for some $\lambda \in \mathbb{R}$.

We now show by contradiction that $\lambda = 0$. If $\lambda < 0$, then we have that $\mathcal{L}^n(C \cap B(0, -\lambda)) = 0$. But then, since $\varphi_{E_j} \to \varphi_C$ in L^1_{loc},

$$\mathcal{L}^n(C \cap B(0, -\lambda) = \lim_{j\to\infty} \mathcal{L}^n(E_{t_j} \cap B(0, -\lambda)) = \lim_{j\to\infty} t_j^{-n}|E \cap B(0, -\lambda(t_j)). \tag{3.27}$$

This is a contradiction to Lemma 3.1. If $\lambda > 0$, then a similar argument yields the same contradiction. Thus, we can conclude that $C = \{x \in \mathbb{R}^n : x_1 < 0\}$, and so $C = T^+(0)$.

Moreover, we have also shown that

$$\int_{B(0,\rho)} |D\varphi_{T^+(0)}| = \lim_{t\to 0} \int_{B(0,\rho)} |D\varphi_{E_t}|. \tag{3.28}$$

Finally, if A is a bounded set such that $H_{n-1}(A \cap T(0)) = 0$, then it is contained in some ball $B(0, \rho)$. Then, Theorem 1.2 immediately yields the result. $\square$

We present an additional result regarding the reduced boundary. This result will help us prove the main result of the next section.

Theorem 3.2. *Suppose $E \subset \mathbb{R}^n$ with $0 \in \partial^* E$. Then, for $\rho, \epsilon > 0$, we define*

$$S_{\rho,\epsilon} = B(0, \rho) \cap \{x : |\langle v(0), x\rangle| < \epsilon\rho. \tag{3.29}$$

Then, the following estimates hold:

$$\lim_{\rho\to 0} \rho^{1-n} \int_{S_{\rho,\epsilon}} |D\varphi_E| = \omega_{n-1}, \tag{3.30}$$

$$\lim_{p\to 0} \rho^{1-n} \mathcal{L}^n(E \cap B(0,\rho) \cap T^-) = 0, \tag{3.31}$$

$$\lim_{p\to 0} \rho^{1-n} \mathcal{L}^n((B(0,\rho)\backslash E) \cap T^+) = 0. \tag{3.32}$$

Remark 3.1. *Intuitively, the previous lemma assets that in small enough balls most of E lies in T^+; conversely, most of $\mathbb{R}^n \backslash E$ lies in T^-. Then, we can see that T somehow splits a small ball $B(0,\rho)$ into two parts corresponding to E and $\mathbb{R}^n \backslash E$.*

4 More on the Reduced Boundary

In this section, we provide more results on the reduced boundary, especially on its regularity. The main result of this section shows that the reduced boundary is, up to a set of $|D\varphi_E|$ measure 0, equivalent to the whole boundary, that the reduced boundary is (up to a set of $|D\varphi_E|$ measure 0) a countable union of C^1 hypersurfaces, and that for any open set Ω,

$$\int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega). \tag{4.1}$$

Such result further strengthens the idea that the reduced boundary is the "continuous" part of the boundary of a Caccioppoli set, and so the properties of such type of sets can be determined by looking at the reduced boundary. In order to prove the main result of this section, we first need an estimate on the measure of subsets of the reduced boundary.

Lemma 4.1. *If E is a Caccioppoli set in $\mathbb{R}^n$ and $B \subset \partial^* E$, then*

$$H_{n-1}(B) \leq 2.3^{n-1} \int_B |D\varphi_E|. \tag{4.2}$$

This lemma allows us to show that if $B \subset \partial^* E$, then, in fact,

$$H_{n-1}(B) = \int_B |D\varphi_E|. \tag{4.3}$$

Now, we introduce a class of sets that will be featured when proving the main result of this section.

Definition 4.1. *Let $H \subset \mathbb{R}^n$. Then, $H \in \Gamma_{n-1}$ if there exists an open set A with $\overline{H} \subset A$ and a C^1 function $f : A \to \mathbb{R}$ with*

$$f(x) = 0 \quad \textit{and} \quad Df(x) \neq 0 \tag{4.4}$$

for all $x \in \overline{H}$.

At first glance, it might be hard to determine whether a set $H \in \Gamma_{n-1}$. The next result solves such issue by providing a very nice characterization of Γ_{n-1}.

Theorem 4.1. *Let $C \subset \mathbb{R}^n$ be compact. Suppose that there exists a vector-valued continuous function $v : C \to \mathbb{R}^n$ with $v \neq 0$ and*

$$\langle v(x), x - y \rangle |x - y|^{-1} \to 0 \tag{4.5}$$

as $|x - y| \to 0$ uniformly for $x, y \in C$. Then, $C \in \Gamma_{n-1}$.

Now, we are ready to present the main result of this section: a theorem first proven by De Giorgi.

Theorem 4.2. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli set. Then,*

$$\partial^* E = \bigcup_{i=1}^{\infty} C_i \cup N, \tag{4.6}$$

where $\int_N |D\varphi_E| = 0$ and each C_i is compact and belongs to Γ_{n-1}. Also, if $B \subset \partial^* E$, then

$$\int_B |D\varphi_E| = H_{n-1}(B). \tag{4.7}$$

Further, if Ω is open, then

$$P(E, \Omega) = \int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega). \tag{4.8}$$

Finally,

$$\overline{\partial^* E} = \partial E. \tag{4.9}$$

Proof. Let $x \in \partial^* E$. Then, up to a rotation and a translation, we may assume that Theorem 3.2 holds for x. Now, let $\{\rho_j\}_{j=1}^{\infty}$ be a sequence of positive numbers that converges to 0. Define

$$f_n(x) = \frac{\mathcal{L}^n(E \cap B(x, \rho) \cap T^-(x))}{\rho_j^n}.$$

Since $\lim_{n\to\infty} f_n(x) = 0$, Egoroff's theorem implies that there exists some set E_i such that $\int_{\partial^* E - E_i} |D\varphi_E| < \frac{1}{2i}$, and $f_n \to 0$ uniformly on $\partial^* E - E_i$. Furthermore, Lusin's theorem implies that there is a compact set C_i with $\int_{F_i - C_i} |D\varphi_E| < \frac{1}{2i}$ such that v restricted to C_i is continuous. Then, letting $N = \partial^* E \backslash \bigcup_{i=1}^{\infty} C_i$, we have that $\partial^* E = \bigcup_{i=1}^{\infty} C_i \cup N$. Also, we have that $\int_N |D\varphi_E| \leq \int_{\partial^* E - C_i} |D\varphi_E| < \frac{1}{i}$ for all i. Hence, $\int_N |D\varphi_E|$. Thus, in order to prove the theorem's first result, we only need to show that each $C_i \in \Gamma_{n-1}$.

Now, consider C_1. By Theorem 3.2, we have that for every ϵ with $0 < \epsilon < 1$, there exists some ω with $0 < \omega < 1$ such that if $\rho < 2\omega$, then

$$\mathcal{L}^n(E \cap B(z,\rho) \cap T^-(z)) < \frac{\epsilon^n \omega_n \rho^n 2^{-n}}{4}, \tag{4.10}$$

and

$$\mathcal{L}^n(E \cap B(z,\rho) \cap T^+(z)) > \frac{\omega_n \rho^n}{2} - \epsilon^n \frac{\omega_n \rho^n}{4} 2^{-n} = \frac{\omega_n \rho^n}{2}(1 - \frac{\epsilon^n}{2^{n+1}}). \tag{4.11}$$

Our aim is to prove that if $x, y \in C_1$ with $|x-y| < \sigma$, then

$$|\langle v(x), x-y\rangle||x-y|^{-1} \leq \epsilon. \tag{4.12}$$

Since ϵ is arbitrary, we will use our alternate characterization of Γ_{n-1} sets given in Theorem 4.1 to conclude that $C_1 \in \Gamma_{n-1}$.

Now, we proceed by contradiction. Suppose that

$$\langle v(x), x-y\rangle|x-y| > \epsilon. \tag{4.13}$$

Then, since $\epsilon < 1$, the triangle inequality implies that

$$B(y, \epsilon|x-y|) \subset T^-(x) \cap B(x, 2|x-y|). \tag{4.14}$$

However, Theorem 3.2 implies that

$$|E \cap B(x, 2|x-y|) \cap T^-(x)| < \frac{\epsilon^n \omega_n |x-y|^n}{4}, \tag{4.15}$$

and that

$$|E \cap B(y, \epsilon|x-y|)| > \frac{\omega_n \epsilon^n |x-y|^n}{4}. \tag{4.16}$$

Yet,

$$|E \cap B(y, \epsilon|x-y|)| \leq |E \cap B(x, 2|x-y|) \cap T^-(x)| < \frac{\epsilon^n \omega_n |x-y|^n}{4}. \tag{4.17}$$

This is a contradiction, and so

$$\langle v(x), x-y\rangle|x-y| \leq \epsilon. \tag{4.18}$$

The same proof can be used to show that

$$\langle v(x), x-y\rangle|x-y| \geq -\epsilon. \tag{4.19}$$

Hence, $C_1 \in \Gamma_{n-1}$, and essentially the same argument implies that $C_i \in \Gamma_{n-1}$ for $i = 2, 3, \ldots$.

Now, we focus on showing that

$$\int_B |D\varphi_E| = H_{n-1}(B) \tag{4.20}$$

for $B \subset \partial^* E$. To do so, we use Lemma 4.1 to get that

$$H_{n-1}(B - C_i) \leq 2.3^{n-1} \int_{B-C_i} |D\varphi_E| < \frac{2.3^{n-1}}{i}. \tag{4.21}$$

Hence, by letting $i \to \infty$, we can assume that $B \subset C_i$; additionally, this would imply that $B \in \Gamma_{n-1}$.

If $B \in \Gamma_{n-1}$, then we have that there exists an open set A containing $\overline{B}$ and a C^1 function $f : A \to \mathbb{R}^n$ such that

$$f = 0 \text{ and } Df \neq 0 \text{ in } \overline{B}. \tag{4.22}$$

Then, the continuity of f implies that

$$V = \{x \in A : f(x) = 0\} \tag{4.23}$$

is a regular hypersurface containing $\overline{B}$. Now, define γ to be the restriction of the H_{n-1} measure to V; that is, $\gamma(F) = H_{n-1}(F \cap V)$ for any H_{n-1} measurable set. Next, since $\overline{B} \subset A$ and A is open, the properties of Hausdorff measures imply that

$$\lim_{\rho \to 0} \rho^{1-n} \gamma(B(x, p)) = \omega_{n-1} \tag{4.24}$$

for $x \in B$. However, the first result of Theorem 3.2 and the fact that $x \in \partial^* E$ imply that

$$\lim_{\rho \to 0} \frac{\gamma(B(x,\rho))}{\int_{B(x,\rho)} |D\varphi_E|} = 1. \tag{4.25}$$

By differentiation of measures, we get that

$$\gamma(B(x,\rho)) = \int_{B(x,\rho) \cap B} |D\varphi_E|. \tag{4.26}$$

And by taking a Vitalli covering of B, we get that

$$H_{n-1}(B) = \int_B |D\varphi_E|. \tag{4.27}$$

Now, the theorem of Besicovitch on differentiation of measures implies that the normal $v(x)$ exists for $|D\varphi_E|$ almost every $x \in \partial E$ and that $|v(x)| = 1$. Hence, $\partial E - \partial^* E$ has $|D\varphi_E|$ measure 0. Thus, we have that

$$P(E,\Omega) = \int_{\Omega} |D\varphi_E| = \int_{\Omega \cap \partial E} |D\varphi_E|$$

$$= \int_{\Omega \cap \partial^* E} |D\varphi_E| + \int_{\Omega \cap (\partial E \setminus \partial^* E)} |D\varphi_E| = \int_{\Omega \cap \partial^* E} |D\varphi_E|. \tag{4.28}$$

Finally, if there exists an open set A with $A \cap \partial^* E = 0$, then our previous computation implies that $\partial E \cap A = 0$. Consequently, $\overline{\partial^* E} = \partial E$. $\square$

We now conclude the section by presenting three more results regarding Caccioppoli sets. The first result gives an explicit representation of the difference between the measure of a ball in an arbitrary Caccioppoli set and the measure of a ball with a slightly different center in such set. The second result gives conditions under which the perturbation of a point in the boundary of a set will actually lie in the set. And the third result gives conditions under which the boundary of a Caccioppoli set is the epigraph of a function, as well as a continuity estimate of such function. These results will be useful when proving the De Giorgi Lemma.

Theorem 4.3. *Let $E \subset \Omega$ be a Caccioppoli set. Fix $\alpha \in \mathbb{R}^n, z \in \Omega, p > 0$. Also, suppose that there exists some $\tau > 0$ such that for every t with $0 < t < \tau$, $B(z + t\alpha, p) \subset \Omega$. Then,*

$$\mathcal{L}^n(E \cap B(z + t\alpha, \rho)) - \mathcal{L}^n(E \cap B(z, \rho)) = \int_0^{\tau} \int_{B(z+t\alpha,\rho)} D_\alpha \varphi_E dx dt. \tag{4.29}$$

Theorem 4.4. *Let $E \subset \Omega$ be a Caccioppoli set. Also, suppose that there exists some $\alpha \in \mathbb{R}^n$ such that*

$$v(x) \cdot \alpha = \lim_{p \to 0} \frac{\int_{B(x,p)} D_\alpha \varphi_E}{\int_{B(x,p)} |D\varphi_E|} \geq q > 0 \tag{4.30}$$

for $|D\varphi_E|$ almost all $x \in \Omega$ and some constant $q < 1$. Then, if $z \in \partial E \cap \Omega$ and $[z, z + k\alpha] \subset \Omega$ for some $k > 0$, then $z + k\alpha$ is interior to E.

Theorem 4.5. *Let $E \subset \Omega$ be a Caccioppoli set, and suppose Ω is open and convex. Also, suppose that*

$$v_n(x) = \lim_{p \to 0} \frac{\int_{B(x,p)} D_n \varphi_E}{\int_{B(x,p)} |D\varphi_E|} \geq q > 0 \tag{4.31}$$

for $|D\varphi_E|$ almost all $x \in \Omega$. Then, there exists some open set $A \subset \mathbb{R}^{n-1}$ and a function $f : A \to \mathbb{R}$ such that

$$\partial E \cap \Omega = \{(y,t) : y \in A, t = f(y)\}. \tag{4.32}$$

Also, f is Lipschitz continuous with Lipschitz constant $\frac{\sqrt{1-q^2}}{q}$.

5 Proving the De Giorgi Lemma: From Harmonic Functions to C^1 sets

In this section, and in the next one, we lay the groundwork for proving the De Giorgi Lemma, the most important step in determining the regularity of minimal sets. Even though we formally prove such lemma in the last section of this thesis, we state the lemma in this section with the purpose of providing perspective on how the proof proceeds. Before, we state the lemma, we present some important definitions.

Definition 5.1. If $f \in BV(\Omega)$, we set

$$v(f,\Omega) = \inf\{\int_\Omega |Dg| : g \in BV(\Omega), spt(g-f) \subset \Omega\}, \tag{5.1}$$

$$\psi(f,\Omega) = \int_\Omega |Df| - v(f,\Omega). \tag{5.2}$$

Additionally, if $f = \varphi_E$ for some $E \subset \Omega$, we write $v(E,\Omega)$ instead of $v(\varphi_E,\Omega)$ and $\psi(E,\Omega)$ instead of $\psi(\varphi_E,\Omega)$. Furthermore, if $\Omega = B(0,\rho)$, we write $v(f,\rho)$ instead of $v(f,B(0,\rho))$ and $\psi(f,\rho)$ instead of $\psi(f,B(0,\rho))$.

From such definition, it can be seen that E is a minimal surface in Ω if and only if $\psi(E,\Omega) = 0$. In light of this fact, we state the De Giorgi Lemma.

Lemma 5.1. *(De Giorgi) For every $n \geq 2$ and for every $\alpha \in (0,1)$, there exists some constant $\sigma(n,\alpha)$ such that, if E is a Caccioppoli set in $\mathbb{R}^n$ and for some $\rho > 0$*

$$\psi(E,\rho) = 0, \tag{5.3}$$

$$\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E| < \sigma(n,\alpha)\rho^{n-1}, \tag{5.4}$$

then in fact

$$\int_{B(0,\alpha\rho)} |D\varphi_E| - |\int_{B(0,\alpha\rho)} D\varphi_E| < \alpha^{n-1}(\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E|).$$

(5.5)

The De Giorgi Lemma is essentially a decay estimate: it specifies that the decay of $\int_{B(0,\alpha\rho)} |D\varphi_E| - |\int_{B(0,\alpha\rho)} D\varphi_E|$ is controlled by α^{n-1}. Such fact plays a key role when proving the regularity of minimal sets. The conditions for the De Giorgi Lemma to hold are that $\psi(E, \rho) = 0$ (i.e., E is a perimeter minimizing set) and that

$$\frac{|\int_{B(0,\rho)} D\varphi_E|}{\int_{B(0,\rho)} |D\varphi_E|} > 1 - \frac{\sigma(n,\alpha)\rho^{n-1}}{\int_{B(0,\rho)} |D\varphi_E|}.$$

Intuitively, the latter condition means that 0 is in the reduced boundary of E.

The right-hand side of the De Giorgi Lemma,

$$Exc(E, \rho) = \rho^{1-n}\{\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E|\}$$

even has a special name: some call it the Excess. Its importance can be observed by noting that, due to the properties of the reduced boundary,

$$Exc(E, \rho) = \rho^{1-n}\{H_{n-1}(B(0,\rho) \cap \partial^* E) - |\int_{B(0,\rho)\cap\partial^* E} v(x) dH_{n-1}|\}.$$

Such representation implies that $|Exc(E, \rho)|$ can be seen as measuring how much the direction of $v(x)$ changes in $B(0, \rho) \cap \partial^* E$. If such quantity is indeed small, then a result like Theorem 4.5 could be applied to give a regularity estimate.

Now, in order to prove the De Giorgi Lemma, we note that if ∂E is the epigraph of a $C^1(A)$ function f, where $A \subset \mathbb{R}^{n-1}$, then we have that

$$\int_{B(0,\rho)} |D\varphi_E| = \int_A \sqrt{1 + |Df|^2}.$$

Also, if $|Df|$ is relatively small, then $\sqrt{1 + |Df|^2}$ will be close to $1 + \frac{|Df|^2}{2}$. This implies that f must almost minimize $\int_A |Df|^2 dx$; in other words, f must almost be a harmonic function.

In order to prove the De Giorgi Lemma, we first prove a similar lemma for harmonic functions. Then, we prove another similar lemma for C^1 functions that are close to harmonic functions and whose gradient is small. Next, we prove another similar lemma again for C^1 sets that are nearly flat.

Afterwards, we prove another similar lemma for Caccioppoli sets that are locally C^1 and flat. In order to reach the proof of the De Giorgi Lemma, we use the estimate from each of the proofs in a progressive order, applying the previous estimate to prove the current one. Thus, we first give a statement for harmonic functions.

Lemma 5.2. *Suppose $u \in C^1(\mathcal{B}(0,\rho))$ is harmonic in $\mathbb{R}^m$ ($\sum_{j=1}^m u_{jj} = 0$). Then, letting*

$$q = \frac{1}{\mathcal{L}^m(\mathcal{B}(0,\rho))} \int_{\mathcal{B}(0,\rho)} Du dx, \tag{5.6}$$

we have that if $\alpha \in (0,1)$, then

$$\int_{\mathcal{B}(0,\alpha\rho)} |Du|^2 - |q|^2 dx \leq \alpha^{m+2} \int_{\mathcal{B}(0,\rho)} |Du|^2 - |q|^2 dx. \tag{5.7}$$

Proof. Since u is harmonic, it can be written as a sum of homogeneous orthogonal harmonic polynomials:

$$u = \sum_{i=0}^{\infty} V_i, \tag{5.8}$$

where each V_i is a harmonic polynomial of degree i, and for $k \neq j$,

$$\int_{\mathcal{B}(0,\alpha\rho)} \langle DV_j, DV_k \rangle dx = \int_{\mathcal{B}(0,\rho)} \langle DV_j, DV_k \rangle dx = 0.$$

Then, the mean-value property of harmonic functions implies that

$$\int_{B(0,\alpha\rho)} DV_j dx = \int_{B(0,\rho)} DV_j dx = DV_j(0) = 0, \tag{5.9}$$

for $j \geq 2$. Consequently, since DV_1 is constant, we get that $q = DV_1$. Hence, applying the orthogonality assumption yields that

$$\int_{\mathcal{B}(0,\rho)} (|Du|^2 - |q|^2) dx = \sum_{j=2}^{\infty} \int_{\mathcal{B}(0,\alpha\rho)} |DV_j|^2 dx, \tag{5.10}$$

$$\int_{\mathcal{B}(0,\alpha\rho)} (|Du|^2 - |q|^2) dx = \sum_{j=2}^{\infty} \int_{\mathcal{B}(0,\alpha\rho)} |DV_j|^2 dx. \tag{5.11}$$

The result now follows from the homogeneity of V_j. $\square$

Now, we take the next step and prove a similar lemma for C^1 functions that are close to harmonic functions (in a sense to be clarified in the estimate).

Lemma 5.3. *Let* $\{\omega_j\}_{j=1}^{\infty}$ *be a sequence of* $C^1(\overline{\mathcal{B}(0,\rho)})$ *functions and let* $\{\beta_j\}_{j=1}^{\infty}$ *be a sequence of positive real numbers. For* $j \in \mathbb{N}$, *let* u_j *be the harmonic function in* $\mathcal{B}(0,\rho)$ *with* $u_j = \omega_j$ *on* $\partial\mathcal{B}(0,\rho)$. *Also, for* $f \in C(B(0,\rho))$ *and* $r \leq \rho$, *let* $\{f\}_r = \frac{1}{\mathcal{L}^n(\mathcal{B}(0,\rho))}\int_{\mathcal{B}(0,r)})f dx$ *(i.e.,* $\{f\}_r$ *is the average of* f *in* $\mathcal{B}(0,\rho)$*). Then, suppose that*

$$\lim_{j\to\infty} \sup_{\mathcal{B}(0,\rho)} |D\omega_j| = 0, \tag{5.12}$$

$$\int_{\mathcal{B}(0,p)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_\rho|^2}\}dx \leq \beta_j, \tag{5.13}$$

$$\limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,p)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{Du_j\}_\rho|^2}\}dx = 0. \tag{5.14}$$

Then, if $\alpha \in (0,1)$,

$$\limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,\alpha p)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2}\}dx \leq \alpha^{m+2}. \tag{5.15}$$

Remark 5.1. *Notice that each* u_j *is unique since solutions of Laplace's equation in bounded domains are known to be unique for smooth boundary data.*

Proof. If we Taylor expand the function $\sqrt{1+x}$ about $x = B^2$, it follows that

$$\sqrt{1+A^2} - \sqrt{1+B^2} - \frac{A^2-B^2}{2\sqrt{1+B^2}} = -\frac{(A^2-B^2)^2}{8(1+t^2)^{\frac{3}{2}}} \tag{5.16}$$

for some t lying between A and B. Thus, we can claim that

$$\sqrt{1+A^2} - \sqrt{1-B^2} \leq \frac{A^2-B^2}{2\sqrt{1+B^2}}. \tag{5.17}$$

Then, if $B^2 < 1$ and assuming $A > B$, we have that

$$\sqrt{1+A^2} - \sqrt{1+B^2} - \frac{A^2-B^2}{2\sqrt{1+B^2}} \geq -\frac{(A^2-B^2)^2}{2\sqrt{1+B^2}}. \tag{5.18}$$

Next, equation 5.18 implies that

$$\int_{\mathcal{B}(0,\alpha\rho)} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2} dx$$

$$\leq \frac{1}{2\sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2}} \int_{\mathcal{B}(0,\alpha\rho)} |D\omega_j|^2 - |\{D\omega_j\}_{\alpha\rho}|^2 dx. \tag{5.19}$$

Now, by definition of $\{D\omega_j\}_{\alpha\rho}$, we have that

$$\int_{\mathcal{B}(0,\alpha\rho)} [|D\omega_j|^2 - |\{D\omega_j\}_{\alpha\rho}|^2] dx$$

$$= \int_{\mathcal{B}(0,\alpha\rho)} |D\omega_j - \{D\omega_j\}_{\alpha\rho}|^2 dx \leq \int_{\mathcal{B}(0,\alpha\rho)} |D\omega_j - \{D\omega_j\}_{\rho}|^2 dx. \tag{5.20}$$

Hence, we have that

$$\limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,\alpha\rho)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2} dx$$

$$\leq \frac{1}{2} \limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,\alpha\rho)} |D\omega_j - \{D\omega_j\}_{\rho}|^2 dx. \tag{5.21}$$

Now, let $A, B, C \in \mathbb{R}^m$. Then, we have that

$$|A-B|^2 \leq (|A-C| + |B-C|)^2 \leq (1+\frac{1}{\epsilon})|A-C|^2 + (1+\epsilon)|B-C|^2 \tag{5.22}$$

for every $\epsilon > 0$. Next, for $\epsilon > 0$ fixed, we get that

$$\int_{B(0,\alpha\rho)} |D\omega_j - \{D\omega_j\}_{\rho}|^2$$

$$\leq (1+\frac{1}{\epsilon}) \int_{B(0,\alpha\rho)} |D\omega_j - Du_j|^2 dx + (1+\epsilon) \int_{B(0,\alpha\rho)} |Du_j - \{D\omega_j\}_{\rho}|^2 dx. \tag{5.23}$$

By the mean-value property of harmonic functions, it follows that

$$\{Du_j\}_{\alpha\rho} = \{Du_j\}_{\rho} = \{D\omega_j\}_{\rho}. \tag{5.24}$$

Then, Lemma 5.2 implies that

$$\int_{\mathcal{B}(0,\alpha\rho)} |Du_j - \{D\omega_j\}_{\rho}|^2 dx \leq \alpha^{m+2} \int_{\mathcal{B}(0,\rho)} |Du_j - \{D\omega_j\}_{\rho}|^2 dx. \tag{5.25}$$

Also, we have that

$$\int_{\mathcal{B}(0,\rho)} |Du_j - \{D\omega_j\}_\rho|^2 dx$$

$$\leq (1+\epsilon)\int_{\mathcal{B}(0,\rho)} |D\omega_j - \{D\omega_j\}_\rho|^2 dx + (1+\frac{1}{\epsilon})\int_{\mathcal{B}(0,\rho)} |D\omega_j - Du_j|^2 dx. \quad (5.26)$$

Consequently, equation 5.24 implies that

$$\int_{\mathcal{B}(0,\alpha\rho)} |D\omega_j - \{D\omega_j\}_\rho|^2 dx$$

$$\leq \alpha^{m+2}(1+\epsilon)^2 \int_{\mathcal{B}(0,\rho)} |D\omega_j - \{D\omega_j\}_\rho|^2 dx + Q(\epsilon,\alpha,m)\int_{\mathcal{B}(0,\rho)} |D\omega_j - Du_j|^2 dx, \quad (5.27)$$

for some constant $Q(\epsilon,\alpha,m)$.

Next, from equation 5.18, we have that

$$\int_{\mathcal{B}(0,\rho)} (\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_\rho|^2})dx \geq$$

$$\frac{1}{2\sqrt{1+|\{D\omega_j\}_\rho|^2}}\{\int_{\mathcal{B}(0,\rho)} (|D\omega_j|^2 - |\{D\omega_j\}_\rho|^2)dx - \int_{\mathcal{B}(0,\rho)} (|D\omega_j|^2 - |\{D\omega_j\}_\rho|^2)^2 dx\}. \quad (5.28)$$

We then notice that

$$(|D\omega_j|^2 - |\{D\omega_j\}_\rho|^2)^2$$

$$\leq |D\omega_j - \{D\omega_j\}_\rho|^2 (\sup_{\mathcal{B}(0,\rho)} |D\omega_j| + |\{D\omega_j\}|_\rho)^2 = m_j |D\omega_j - \{D\omega_j\}_\rho|^2,$$

where m_j denotes a term with $m_j \to 0$ as $j \to \infty$. Hence, we have that

$$\int_{\mathcal{B}(0,\rho)} (\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_\rho|^2})dx$$

$$\geq \frac{1-m_j}{2\sqrt{1+|\{D\omega_j\}_\rho|^2}} \int_{\mathcal{B}(0,\rho)} (|D\omega_j|^2 - |\{D\omega_j\}_\rho|^2)dx. \quad (5.29)$$

Then, equations 5.29, 5.12, and 5.13 imply that

$$\limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,\rho)} (|D\omega_j|^2 - |\{D\omega_j\}_\rho|^2)dx \leq 2. \quad (5.30)$$

Now, we may use equations 5.29 and 5.17 to get that

$$\int_{\mathcal{B}(0,\rho)} (|\{D\omega_j\}_\rho|^2 - |Du_j|^2)dx$$

$$\leq 2\sqrt{1+|\{D\omega_j\}_\rho|^2}\int_{\mathcal{B}(0,\rho)} \sqrt{1+|\{D\omega_j\}_\rho|^2} - \sqrt{1+|Du_j|^2}dx. \tag{5.31}$$

Since u is harmonic and $\omega_j = u_j$ on $\partial\mathcal{B}(0,\rho)$, integration by parts yields that

$$\int_{\mathcal{B}(0,\rho)} \langle D\omega_j, Du_j\rangle dx = \int_{\mathcal{B}(0,\rho)} |Du_j|^2 dx, \tag{5.32}$$

and thus,

$$\int_{\mathcal{B}(0,\rho)} |D\omega_j - Du_j|^2 dx = \int_{\mathcal{B}(0,\rho)} |D\omega_j|^2 - |Du_j|^2 dx. \tag{5.33}$$

As a result, adding and subtracting $\{D\omega_j\}_\rho$ from equation 5.33 reveals that

$$\int_{\mathcal{B}(0,\rho)} |D\omega_j - Du_j|^2 dx \leq$$

$$2\sqrt{1+|\{D\omega_j\}_\rho|^2}[(\int_{\mathcal{B}(0,\rho)} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|Du_j|^2})dx \tag{5.34}$$

$$+\frac{m_j}{1-m_j}\int_{\mathcal{B}(0,\rho)} \sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_\rho|^2})dx].$$

Thus, we have that

$$\lim_{j\to\infty} \beta_j^{-1}\int_{\mathcal{B}(0,\rho)} |D\omega_j - Du_j|^2 dx = 0. \tag{5.35}$$

Hence, it follows that

$$\limsup_{j\to\infty} \beta_j^{-1}\int_{\mathcal{B}(0,\alpha\rho)} [\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2}]dx \leq (1+\epsilon^2)\alpha^{m+2}. \tag{5.36}$$

Letting $\epsilon \to 0$, we obtain the result. □

The following lemma is similar to the previous one, except that now we look at sets whose boundary is the graph of a C^1 function with small gradients. Intuitively, this means that the boundary of the set will be flat. The conditions of the lemma are modified slightly from the previous one to account for the fact that we are now working with epigraphs of functions instead of functions. To such end, we define

$$W = \{(x,t) : x \in \mathcal{B}(0,\rho), t < \omega(x)\} \tag{5.37}$$

$$Q = \{(x,t) : x \in \mathcal{B}(0,\rho), \min \omega - 1 < t < \max \omega + 1\} \tag{5.38}$$

for $\omega \in C^1(\mathcal{B}(0,\rho))$.

Lemma 5.4. *Let $\{\omega\}_{j=1}^{\infty}$ be a sequence of $C^1(B(0,\rho))$ functions and let $\{\beta_j\}_{j=1}^{\infty}$ be a sequence of positive real numbers. Suppose that*

$$\lim_{j\to\infty} \sup_{\mathcal{B}(0,\rho)} |D\omega_j| = 0, \tag{5.39}$$

$$\int_{\mathcal{B}(0,p)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_\rho|^2}\}dx \le \beta_j, \tag{5.40}$$

$$\lim_{j\to\infty} \beta_j^{-1}\psi(W_j, Q_j) = 0. \tag{5.41}$$

Then, if $\alpha \in (0,1)$,

$$\limsup_{j\to\infty} \beta_j^{-1} \int_{\mathcal{B}(0,\alpha\rho)} \{\sqrt{1+|D\omega_j|^2} - \sqrt{1+|\{D\omega_j\}_{\alpha\rho}|^2}\}dx \le \alpha^{m+2}. \tag{5.42}$$

Proof. Let u_j be the harmonic function in $\mathcal{B}(0,\rho)$ that equals ω_j on $\partial\mathcal{B}(0,\rho)$. Then, by definition of $\psi(W_j, Q_j)$, it follows that

$$\int_{\mathcal{B}(0,\rho)} (\sqrt{1+|D\omega_j|^2} - \sqrt{1+|Du_j|^2})dx \le \psi(W_j, Q_j).$$

Hence, Lemma 5.3 immediately yields the result. □

In the next lemma, we make the transition from C^1 sets whose boundary is the graph of a function to Caccioppoli sets that are locally C^1 sets. This lemma uses the previous one with $m = n - 1$.

Lemma 5.5. *Suppose that* $\{L_j\}_{j=1}^{\infty}$ *is a sequence of Caccioppoli sets in* $\mathbb{R}^m$ *and that* $\{\beta_j\}_{j=1}^{\infty}$ *is a sequence of positive numbers. Then, if there exists* $\rho > 0$ *such that*

$$\int_{B(0,\rho)} |D\varphi_{L_j}| - |\int_{B(0,\rho)} D\varphi_{L_j}| \leq \beta_j, \tag{5.43}$$

$$\partial L_j \cap B(0,\rho) \text{ is a } C^1\text{-hypersurface}, \tag{5.44}$$

$$\lim_{j\to\infty} \inf_{\partial L_j \cap B(0,\rho)} v_n^j(x) = 1 \tag{5.45}$$

(where $v^j(x)$ *is the normal to* L_j *at the point* x*), and*

$$\lim_{j\to\infty} B_j^{-1}\psi(L_j,\rho) = 0, \tag{5.46}$$

then, for $\alpha \in (0,1)$*,*

$$\limsup_{j\to\infty} \beta_j^{-1}\{\int_{B(0,\alpha\rho)} |D\varphi_{L_j}| - |\int_{B(0,\alpha\rho)} D\varphi_{L_j}|\} \leq \alpha^{n+1}. \tag{5.47}$$

6 Proving the De Giorgi Lemma: From C^1 Sets to Arbitrary Caccioppoli sets

In this section, we approximate Caccioppoli minimal sets whose characteristic function has distributional derivatives that all lie approximately in the same direction. To do so, we will approximate such sets with C^1 sets that are nearly flat, so that we may use the theory developed in the previous section. However, we cannot use the mollified functions from the first chapter, as they do not yield the desired results. We must then come up with a new way of mollifying functions. To such end, we define

$$\eta_\epsilon(x) = \frac{n+1}{\omega_n}\epsilon^{-n} \max\{(1 - \frac{|x|}{\epsilon}, 0\}. \tag{6.1}$$

These functions, although only Lipschitz continuous and having support in $\overline{B(0,1)}$ rather than in $B(0,1)$ do satisfy many of the properties of mollifiers. Additionally, they satisfy other properties that help us prove the De Giorgi Lemma.

Throughout the chapter, we denote $f_\epsilon(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)f(y)dy$. First, we prove a technical lemma regarding $(\varphi_E)_\epsilon$.

Lemma 6.1. *Let E be a Borel set $\epsilon > 0$. Then, $\varphi_\epsilon := (\varphi_E)_\epsilon \in C^1(\mathbb{R}^n)$. Also, if $x \in \mathbb{R}^n$ and $\rho < \frac{1}{n}$ satisfy*

$$n^2\rho^2 < \varphi_\epsilon(x) < 1 - n^2\rho^2, \tag{6.2}$$

then

$$dist(x, \partial E) \leq (1-\rho)\epsilon. \tag{6.3}$$

In the following lemma, we prove additional properties of the mollified functions used in this section. Such properties essentially imply that the mollification process is in some way continuous.

Lemma 6.2. *Suppose $f \in BV(B(0,1))$, and that $\tau + \epsilon \leq 1$. Then,*

$$\int_{B(0,\tau)} |f_\epsilon - f|dx \leq \epsilon \int_{B(0,\tau+\epsilon)} |Df|, \tag{6.4}$$

$$\int_{B(0,\tau)} |Df_\epsilon| - \int_{B(0,\tau)} |Df| \leq \int_{B(0,\tau+\epsilon)-B(0,\epsilon)} |Df|. \tag{6.5}$$

The following theorem is a very important result because it shows that if we have a minimal set such that the derivatives of its characteristic function lie in almost the same direction, then the mollified functions of the previous lemmas have derivatives that lie in approximately the same direction. We make this notion more precise in the statement of the theorem.

Theorem 6.1. *Let E be a Caccioppoli set and $\gamma > 0$ such that*

$$\psi(E, 1) = 0, \tag{6.6}$$

$$\int_{B(0,1)} (|D\varphi_E| - D_n(\varphi_E)) \leq \gamma. \tag{6.7}$$

Then, for each integer p there exists some $\lambda(\gamma, p)$ that converges to 0 as γ converges to 0, with the property that if $\epsilon = \gamma^p$ and

$$\varphi(x) = (\varphi_E)_\epsilon,$$

then

$$\inf\{\frac{D_n(\varphi(x))}{|D\varphi(x)|} : |x| < 1 - 2\gamma^{\frac{1}{2(n-1)}} \text{ and } n^2\gamma^2 < \varphi(x) < 1 - n^2\gamma^2\} > 1. \tag{6.8}$$

With the aim of proving a De Giorgi-type lemma for sequences of Caccioppoli sets, we now show that sequences of C^1 sets approximating Caccioppoli sets satisfy conditions like those described in the previous section. Afterwards, proving the desired De Giorgi-type lemma will be relatively easy.

Lemma 6.3. *Let $\{E_j\}$ be a sequence of Caccioppoli sets such that*

$$\psi(E_j, 1) = 0, \tag{6.9}$$

$$\int_{B(0,1)} |D\varphi_{E_j}| - \int_{B(0,1)} D_n\varphi_{E_j} \leq \gamma_j, \tag{6.10}$$

for $\sum_{i=1}^{\infty} \gamma_j < \infty$. Then, for a.e. $t \in (0,1)$ there exists a sequence of sets $\{L_j\}$ such that

$$\lim_{j\to\infty} \gamma_j^{-1}\psi(L_j, t) = 0, \tag{6.11}$$

$$\lim_{j\to\infty} \gamma_j^{-1}\{\int_{B(0,t)} |D\varphi_{L_j}| - \int_{B(0,t)} |D\varphi_{E_j}|\} = 0, \tag{6.12}$$

$$\lim_{j\to\infty} \gamma_j^{-1} \left| \int_{B(0,t)} D\varphi_{L_j} - \int_{B(0,t)} D\varphi_{E_j} \right| = 0, \tag{6.13}$$

$$\partial L_j \cap B(0,t) \text{ is a } C^1 \text{ hypersurface}, \tag{6.14}$$

and, for $s < t$

$$\liminf_{j\to\infty} \{v_n^j(x) : |x| < s, x \in \partial L_j\} = 1. \tag{6.15}$$

Remark 6.1. *In order to prove Lemma 6.3, we need some auxiliary results. First, if E is a Caccioppoli set, then we have that*

$$\int_{B(0,1)} |D\varphi_E| \leq \frac{1}{2} n\omega_n r^{n-1}. \tag{6.16}$$

Also, if $f, g \in BV(B(0,R))$ and $\rho < R$, then

$$|v(f,\rho) - v(g,\rho)| \leq \int_{\partial B(0,\rho)} |f^- - g^-| dH_{n-1}. \tag{6.17}$$

Further, if $\psi(f,R) = 0$ and $g \in BV(B(0,\rho))$,

$$\int_{B(0,\rho)} |Df| \leq \int_{B(0,\rho)} |Dg| + \int_{\partial B(0,\rho)} |f^- - g^-| dH_{n-1}, \tag{6.18}$$

where f^- denotes the trace as approximated from the inside of the set. We note that if $f = \varphi_E$ and $g = \varphi_L$ for some Caccioppoli sets E and L, then $f^- = f = \varphi_E$ and $g^- = g = \varphi_L$ up to an H_{n-1} null set.

Additionally, if Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and $f \in BV(\Omega)$, then there exists a function $\phi \in L^1(\partial\Omega)$ such that for H_{n-1}-almost all $x \in \partial\Omega$,

$$\lim_{\rho\to 0} \int_{B(x,\rho)\cap\Omega} |f(z) - \phi(x)| dz = 0. \tag{6.19}$$

Moreover, if $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_\Omega f \boldsymbol{\nabla} \cdot g dx = -\int_\Omega \langle g, Df \rangle + \int_{\partial\Omega} \phi \langle g, n \rangle dH_{n-1} \tag{6.20}$$

where n is the unit outer normal. The last result implies that if $\Omega = B(0,1)$ and we take a sequence of compactly supported vector fields converging to the vector field $(1, ..., 1)$ (n - times), then a passage to the limit reveals that for a.e. $t \in (0,1)$

$$\int_{B(0,t)} Df = \int_{\partial B(0,t)} f \frac{x}{|x|} dH_{n-1}. \tag{6.21}$$

Proof. Set $\epsilon_j = \gamma_j^4$. Then, Lemma 6.1 and Remark 5.1 imply that if $f_j = (\varphi_{E_j})_{\epsilon_j}$ and $\tau \in (0,1)$, then

$$\limsup_{j\to\infty} \gamma_j^{-4} \int_{B(0,\tau)} |f_j - \varphi_{E_j}| dx \le \int_{B(0,\tau)} |D\varphi_{E_j}| \le \frac{1}{2} n\omega_n r^{n-1} \le \frac{1}{2} n\omega_n. \tag{6.22}$$

Consequently, we see that

$$\gamma_j^{-3} \int_{B(0,\tau)} |f_j - \varphi_{E_j}| dx \to 0 \tag{6.23}$$

as $j \to \infty$. Then, since $\int_{B(0,\tau)} |f_j - \varphi_{E_j}| = \int_0^1 dt \int_{\partial B(0,t)} |f_j - \varphi_{E_j}| dH_{n-1}$, we conclude that there exists a set N_1 of $\mathcal{L}^1$ measure 0 in $(0,1)$ such that for $t \in (0,1)\backslash N_1$,

$$\gamma_j^{-3} \int_{\partial B(0,t)} |f_j - \varphi_{E_j}| dH_{n-1} \to 0. \tag{6.24}$$

Now, define μ to be a measure given by $\mu(\Omega) = \sum_{j=1}^\infty \gamma_j \int_\Omega |D\varphi_{E_j}|$. Since $\sum_{j=1}^\infty \gamma_j < \infty$, Remark 5.1 and Lemma 6.1 imply that

$$\int_{B(0,1)} d\mu < \infty. \tag{6.25}$$

Then, since $\int_{B(0,t)} d\mu$ is a monotonically increasing function of t, it is differentiable almost everywhere. As a result, we may conclude that there exists a set N_2 of $\mathcal{L}^1$ measure 0 such that

$$\lim_{j\to\infty} \gamma_j^{-4} \int_{B(0,t+\gamma_j^4)\backslash B(0,t)} d\mu < \infty. \tag{6.26}$$

This then implies by our definition of μ that

$$\limsup_{j\to\infty} \gamma_j^{-3} \int_{B(0,t+\gamma_j^4)\backslash B(0,t)} d\mu < \infty. \tag{6.27}$$

Next, since $\gamma_j \to 0$ as $j \to \infty$, we can use Lemma 6.1 to get that

$$\limsup_{j\to\infty} \gamma_j^{-2} \int_{B(0,t+\gamma_j^{-4})} |D\varphi_{E_j}| \le 0. \tag{6.28}$$

As a consequence, Lemma 6.1 implies that, if $t \in (0,1)\backslash N_2$, then

$$\limsup_{j\to\infty} \gamma_j^{-2}\{\int_{B(0,t)} |Df_j| - \int_{B(0,t)} |D\varphi_{E_j}|\} \leq 0. \tag{6.29}$$

Now, let $S_j(\theta)$ denote the super sets of f_j in $B(0,1)$, that is, $S_j(\theta) = \{x \in B(0,1) : f_j(x) > \theta\}$. Then, the layer-cake principle implies that

$$\int_{B(0,t)} |Df_j| = \int_0^1 d\theta \int_{B(t,0)} |D\varphi_{S_j(\theta)}| \geq \int_{n^2\gamma_j^2}^{1-n^2\gamma_j^2} \int_{B(t,0)} |D\varphi_{S_j(\theta)}|. \tag{6.30}$$

Hence, it follows that there is some $\theta_j \in (n^2\gamma_j^2, 1 - n^2\gamma_j^2)$ with the property that

$$(1 - 2n^2\gamma_j^2) \int_{B(0,t)} |D\varphi_{S_j(\theta_j)}| \leq \int_{B(0,t)} |Df_j|. \tag{6.31}$$

Now, by the properties of f_j, $L_j := \partial S_j(\theta_j)$ is regular. Moreover, since $|D\varphi_{S_j(\theta_j)}|$ takes its support on L_j, equations 6.29 and 6.31 imply that

$$\limsup \gamma_j^{-1}{}_{j\to\infty}\{\int_{B(0,t)} |D\varphi_{L_j}| - \int_{B(0,t)} |D\varphi_{E_j}|\} \leq 0. \tag{6.32}$$

Then, Lemma 1.2 implies that if $t \in (0,1)$, then

$$\int_{\partial B(0,t)} |\varphi_{L_j} - \varphi_{E_j}| dH_{n-1} \leq n^{-2}\gamma_j^{-2} \int_{\partial B(0,t)} |f_j - \varphi_{E_j}| dH_{n-1}. \tag{6.33}$$

Next, equation 6.24 implies that if $t \in (0,1)\backslash N_2$, then

$$\lim_{j\infty} \gamma_j^{-1} \int_{\partial B(0,t)} |\varphi_{L_j} - \varphi_{E_j}| dH_{n-1} = 0. \tag{6.34}$$

Then, equation 6.18 along with 6.29 imply that for $t \in (0,1)\backslash N_1$,

$$\limsup_{j\to\infty} \gamma_j^{-1}\{\int_{B(0,t)} |D\varphi_{E_j}| - \int_{B(0,t)} |D\varphi_{L_j}|\} \leq 0. \tag{6.35}$$

Such result along with equation 6.32 yields equation 6.12.

Equation 6.13 follows equation 6.21, equation 6.34 and an application of the triangle inequality.

In order to prove 6.11, we notice that

$$|\psi(L_j,t)-\psi(E_j,t)| \leq |\int_{B(0,t)} D\varphi_{L_j}-\int_{B(0,t)} D\varphi_{E_j}+v(L_j,t)-v(E_j,t)|. \quad (6.36)$$

Hence, 6.11 follows by applying the triangle inequality to the previous equation and using equations 6.12, 6.34, and 6.17.

Next, since ∂L_j corresponds to the graph of f_j and f_j is at least Lipschitz continuous, equation 6.14 follows. Additionally, we notice that E_j satisfies the assumptions of theorem 6.1. Thus, 6.15 follows from the conclusion of theorem 6.1 by taking $\gamma = \gamma_j$ and passing to the limit. □

Finally, we prove the De Giorgi-type lemma for sequences of Caccioppoli sets. This will be the last result needed to prove the De Giorgi Lemma.

Theorem 6.2. *Suppose $\{E_j\}_{j=1}^{\infty}$ is a sequence of Caccioppoli sets such that*

$$\psi(E_j,1)=0, \quad (6.37)$$

$$\int_{B(0,1)} |D\varphi_{E_j}| - \int_{B(0,1)} D_n\varphi_{E_j} \leq \gamma_j, \quad (6.38)$$

$$\sum_{j=1}^{\infty} \gamma_j < \infty. \quad (6.39)$$

Then, if $\alpha \in (0,1)$, we have that

$$\limsup_{j\to\infty} \gamma_j^{-1}\{\int_{B(0,\alpha)} |D\varphi_{E_j}| - |\int_{B(0,\alpha)} D\varphi_{E_j}|\} \leq \alpha^{n+1}. \quad (6.40)$$

Proof. Since the sequence of Caccioppoli sets satisfies the hypotheses of the previous theorem, we may consider its associated sequence of sets $\{L_j\}_{j=1}^{\infty}$. Next, assume that the inequalities hold for a sequence $\{t_j\}_{j=1}^{\infty}$, with each $t_j \in (0,1)$ and $\lim_{j\to\infty} t_j = 1$. Additionally, let $\{s_j\}_{j=1}^{\infty}$ be such that $t_{j-1} < s_j < t_j$. Then, we have that each L_j satisfies the hypotheses of Lemma 12 for each s_j and for any sequence $\{\beta_j\}_{j=1}^{\infty}$ such that with $\beta_j < \gamma_j$ and $\lim_{j\to\infty} \frac{\beta_j}{\gamma_j} = 1$. As a result, Lemma 5.4 implies that

$$\limsup_{j\to\infty} \gamma_j^{-1}\{\int_{B(0,t_j(\frac{\alpha}{s_j}))} |D\varphi_E| - |\int_{B(0,t_j(\frac{\alpha}{s_j}))} D\varphi_E|\} \leq (\frac{\alpha}{s_j})^{n+1}. \quad (6.41)$$

Letting $j \to \infty$, we have that

$$\limsup_{j\to\infty} \gamma_j^{-1}\{\int_{B(0,\alpha)} |D\varphi_{E_j}| - |\int_{B(0,\alpha} D\varphi_{E_j}|\} \leq \alpha^{n+1}. \tag{6.42}$$

□

7 The De Giorgi Lemma and its Corollaries

We now prove the De Giorgi Lemma. We use the De Giorgi Lemma to show that the reduced boundary of a minimal set is locally analytic. Even though singularities may occur in the complement of the reduced boundary, the De Giorgi Lemma may be used to derive the fact that the $n-1$ Hausdorff measure of the singular set $\partial E \backslash \partial^* E$ is 0.

Lemma 7.1. *(De Giorgi) For every $n \geq 2$ and for every $\alpha \in (0,1)$, there exists some constant $\sigma(n,\alpha)$ such that, if E is a Caccioppoli set in $\mathbb{R}^n$ and for some $\rho > 0$,*

$$\psi(E,\rho) = 0, \tag{7.1}$$

$$\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E| < \sigma(n,\alpha)\rho^{n-1}, \tag{7.2}$$

then, in fact,

$$\int_{B(0,\alpha\rho)} |D\varphi_E| - |\int_{B(0,\alpha\rho)} D\varphi_E| < \alpha^{n-1}(\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E|). \tag{7.3}$$

Proof. Assume, by way of contradiction, that the theorem is not true. Then, we can find a sequence of Caccioppoli sets $\{F_j\}_{j=1}^{\infty}$, a sequence of points $\{x_j\}_{j=1}^{\infty}$ in $\mathbb{R}^n$, a sequence $\{\rho_j\}_{j=1}^{\infty}$ in $\mathbb{R}$, and a sequence $\{\sigma_j\}_{j=1}^{\infty}$ in $\mathbb{R}$ such that if we define $\int_{B(0,\rho)} |D\varphi_E| - |\int_{B(0,\rho)} D\varphi_E| = \gamma_j \rho_j^{n-1}$, then

$$\psi(F_j,\rho_j) = 0, \tag{7.4}$$

$$\int_{B(x_j,\rho_j)} |D\varphi_{F_j}| - \int_{B(x_j,\rho_j)} D_n\varphi_{F_j} = \gamma_j \rho_j^{n-1}, \tag{7.5}$$

$$\sum_{j=1}^{\infty} \gamma_j < \infty, \tag{7.6}$$

and

$$\int_{B(x_j,\alpha\rho_j)} |D\varphi_{F_j}| - |\int_{B(x_j,\alpha\rho_j)} D\varphi_{F_j}| > \alpha^n \gamma_j \rho_j^{n-1}.$$

Then, for each j, we apply a rotation so that x_j is rotated to the origin, then another rotation so that the vector $\int_{B(0,\rho)} D\varphi_{F_j}$ is parallel with the x_j axis, and then we preform a dilation of ratio ρ_j. Next, letting the rotated sets be E_j, we find that

$$\psi(E_j, 1) = 0,$$

$$\int_{B(0,1)} |D\varphi_{E_j}| - \int_{B(0,1)} D_n\varphi_{E_j} = \gamma_j$$

$$\int_{B(0,\alpha)} |D\varphi_{E_j}| - |\int_{B(0,\alpha)} D\varphi_{E_j}| > \alpha^n \gamma_j.$$

The sequence of sets $\{E_j\}$ is in direct contradiction of Theorem 6.2. Hence, the De Giorgi Lemma must be true. □

With the De Giorgi Lemma in hand, we are finally able to show that the reduced boundary of minimal sets is locally analytic. Before doing so, we need an auxiliary result that is a consequence of the De Giorgi Lemma.

Theorem 7.1. *Let $x \in \partial E$ for some Caccioppoli set E, and suppose that the conditions of the De Giorgi Lemma hold. Then, if $0 < s < t < \rho$, then*

$$|v_s(x) - v_t(x)| \leq \eta(\alpha, n)\sqrt{\frac{t}{\rho}}, \tag{7.7}$$

where

$$\eta(\alpha, n) = \frac{2 - \sqrt{\alpha}}{1 - \sqrt{\alpha}}\sqrt{\frac{\sigma(n, \alpha)}{\omega_{n-1}\alpha^n}}. \tag{7.8}$$

Remark 7.1. *Before proving such theorem, we need the following result. If E is a Caccioppoli set with $x \in \partial^* E$, then*

$$\omega_{n-1} r^{n-1} \leq \int_{B(x,r)} |D\varphi_E|. \tag{7.9}$$

Proof. By applying a translation and a dilation, we may suppose that $x = 0$ and that $\rho = 1$.

First, we work on the special case where $t = \alpha^j$ and $s = \beta\alpha^j$ for some j and $\alpha \leq \beta < 1$. Next, we define

$$u_j = \frac{\int_{B(0,\alpha^j)} D\varphi_E}{\int_{B(0,\alpha^j)} |D\varphi_E|},$$

$$v_j = \frac{\int_{B(0,\beta\alpha^j)} D\varphi_E}{\int_{B(0,\beta\alpha^j)} |D\varphi_E|},$$

$$m_j = \int_{B(0,\alpha^j)} |D\varphi_E|,$$

$$\mu_j = \int_{B(0,\beta\alpha^j)} |D\varphi_E|.$$

Then, we use the facts that $|u_j| \leq 1$ and $|v_j| \leq 1$ along with an application of the parallelogram law to get that

$$|u_j - v_j| \leq \sqrt{2}\sqrt{1 - \langle u_j, v_j \rangle}.$$

Now, we estimate that

$$(1 - \langle u_j, v_j \rangle)\mu_j = \int_{B(0,\beta\alpha^j)} (|D\varphi_E| - \langle u_j, D\varphi_E \rangle)$$

$$\leq \int_{B(0,\alpha^j)} (|D\varphi_E| - \langle u_j, D\varphi_E \rangle) = m_j(1 - |u_j|^2) \leq 2m_j(1 - |u_j|).$$

Next, by iterating the De Giorgi Lemma j times, we get that

$$m_j(1 - |u_j|) \leq \alpha^{nj}\sigma(n, \alpha).$$

As a consequence,

$$|u_j - v_j| \leq 2\sqrt{\sigma(n, \alpha)} \left(\frac{\alpha^{nj}}{\mu_j}\right).$$

Then, since we have that $0 \in \partial E$, equation 7.9 implies that

$$\mu_j \geq \omega_{n-1}(\alpha^j\beta)^{n-1} \geq \omega_{n-1}\alpha^{(j+1)(n-1)}.$$

Hence, we conclude that

$$|u_j - v_j| \leq 2\sqrt{\frac{\sigma(n, \alpha)}{\omega_{n-1}\alpha^n}}\sqrt{\alpha^{j+1}}.$$

We have thus proven the theorem for the special case $t = \alpha^j$ and $s = \beta\alpha^j$.

Now, let $s, t \in (0, 1)$ with $s < t$. Then, choose integers j and k with the property that

$$\alpha^{j+1} < t \leq \alpha^j, \alpha^{j+k} \leq s < \alpha^{j+k-1}.$$

Next, we use the triangle inequality and the De Giorgi Lemma to obtain the following estimate:

$$|v_t - v_s| \leq |v_t - u_j| + \sum_{j=0}^{k+2} |u_{j+1} - u_{j+i+1}| + |u_{j+k-1} - v_s|$$

$$\leq 2\sqrt{\frac{\sigma(n,\alpha)}{\omega_{n-1}\alpha^n}} \left(\sqrt{\alpha^{j+1}} + \sum_{i=0}^{\infty} \sqrt{\alpha^{j+i+1}} \right) = \frac{2-\sqrt{\alpha}}{1-\sqrt{\alpha}} \sqrt{\frac{\sigma(n,\alpha)}{\omega_{n-1}\alpha^n}} \sqrt{\alpha^{j+1}}$$

$$\leq \frac{2-\sqrt{\alpha}}{1-\sqrt{\alpha}} \sqrt{\frac{\sigma(n,\alpha)}{\omega_{n-1}\alpha^n}} \sqrt{t}.$$

The result is thus proven in the general case. □

The result also yields the following important corollary.

Corollary 7.1. *Suppose that x satisfies the hypotheses of Theorem 21. Then, $x \in \partial^* E$ and*

$$|v(x) - v_t(x)| \leq \eta(n,\alpha) \sqrt{\frac{t}{\rho}}. \tag{7.10}$$

Now, we may finally prove the regularity of the reduced boundary of perimeter minimizing sets using the De Giorgi Lemma and some facts from the theory of elliptic partial differential equations.

Theorem 7.2. *(Regularity of Minimal Sets) Let $E \subset \mathbb{R}^n$ be a Caccioppoli set and $x \in \partial E$. Suppose $\rho > 0$ and $\alpha \in (0,1)$ are such that*

$$\psi(E, B(x,\rho)) = 0, \tag{7.11}$$

$$\int_{B(x,\rho)} |D\varphi_E| - \left| \int_{B(x,\rho)} D\varphi_E \right| \leq \sigma(n,\alpha)\rho^{n-1}. \tag{7.12}$$

Then, $\partial E \cap B(x,r)$ is an analytic hypersuface for $r = \rho(\alpha - \alpha^{\frac{n}{n-1}})$.

Remark 7.2. *Before proving such theorem, we need the following auxiliary result. If E is a Caccioppoli set in Ω with the property that $v(x)$ exists for every $x \in \partial E \cap \Omega$, then $\partial E \cap \Omega$ is a C^1 hypersurface.*

Proof. Suppose that $z \in \partial E \cap B(x,r)$. Then, set $R = \rho\alpha^{\frac{n}{n-1}}$. Next, an application of the triangle inequality yields that $B(z,R) \subset B(x,\alpha\rho)$. The De Giorgi Lemma implies that

$$\int_{B(z,R)} |D\varphi_E| - |\int_{B(z,R)} D\varphi_E|$$

$$\leq \int_{B(x,\alpha\rho)} |D\varphi_E| - |\int_{B(x,\alpha\rho)} D\varphi_E| \leq \sigma(n,\alpha)\alpha^n\rho^{n-1}. \tag{7.13}$$

Consequently, we see that

$$\int_{B(z,R)} |D\varphi_E| - |\int_{B(z,R)} D\varphi_E| \leq \sigma(n,\alpha)R^{n-1}. \tag{7.14}$$

Hence, Corollary 7.1 implies that $z \in \partial^* E$. Then, we have that

$$|v(z) - \frac{\int_0^t ds \int_{B(z,s)} D\varphi_E}{\int_0^t ds \int_{B(z,s)} |D\varphi_E|}| \leq \eta(n,\alpha)\sqrt{\frac{t}{R}}. \tag{7.15}$$

Next, if we define

$$f(t,z) = \frac{\int_0^t ds \int_{B(z,s)} D\varphi_E}{\int_0^t ds \int_{B(z,s)} |D\varphi_E|}, \tag{7.16}$$

then not only is such function continuous, but we also have that $\lim_{t\to 0} f(t,z) = v(z)$ uniformly for $z \in B(x,r)$. Consequently, $v(z)$ is continuous on $\partial E \cap B(x,r)$. Then, we may use remark 7.2 to get that $\partial E \cap B(x,r)$ is a C^1 hypersurface.

Since $\partial E \cap B(x,r)$ is a C^1 hypersurface, it is the graph of a function that minimizes the energy functional $\int \sqrt{1+|Df|^2}$. By the theory of elliptic partial differential equations, such a function is in fact analytic. Thus, we have proven that $\partial E \cap B(x,r)$ is an analytic hypersurface. □

Additionally, we also have that the set of points in the boundary of a minimal set not satisfying the assumptions of Theorem 7.2 has Hausdorff measure 0.

Theorem 7.3. *Let $E \subset \mathbb{R}^n$ be a Caccioppoli set satisfying*

$$\psi(E,1) = 0. \tag{7.17}$$

Then,

$$H_{n-1}(\partial E - \partial^* E) = 0. \tag{7.18}$$

Theorems 7.2 and 7.3 provide a regularity result for perimeter minimizing sets. Both of them are essentially corollaries of the De Giorgi Lemma, which highlights the importance of such lemma.

8 Discussion

The results demonstrate that there always exist perimeter minimizing sets within a given bounded domain, proving the existence of perimeter minimizing sets. By showing their existence, the regularity properties of perimeter minimizing sets may be studied. The De Giorgi Lemma shows conditions under which the decay of the perimeter around a point of the boundary can be controlled. This lemma plays an important role when proving that the reduced boundary of a minimal set is analytic and is thus basic to this project, as it determines the regularity of the reduced boundary of a perimeter minimizing set. The work uses tools of geometric measure theory to prove the De Giorgi Lemma, a fundamental resource in the study of minimal sets.

Bibliography

[1] E. Giusti. *Minimal surfaces and functions of bounded variation.* Birkhauser, 1984.

[2] E. De Giorgi. Su una teoria generale della misura (r - 1)-dimensionale in uno spazio ad r dimensioni. Ann. Mat. Pura Appl. (4) 36 (1954) 191-213.

[3] E. De Giorgi. Nuovi teoremi relativi alle misure (r - 1)-dimensionale inn uno spazio ad r dimensioni. Ricerche Mat. 4 (1955), 95-113.

[4] E. De Giorgi. Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari. Mem. Acc. Sci. Torino 3 (1957) 25-43.

[5] E. De Giorgi. Complementi alla teoria della misura (n - 1)-dimensionale in uno spazio n-dimensionale. Sem. Mat. Scuola Norm. Sup Pisa, 1960-61. Editrice Tecnico Scientifica, Pisa, 1961.

[6] E. De Giorgi. Frontiere orientate di misura minima. Sem. Mat. Scuola Norm. Sup. Pisa, 1960-61. Editrice Tecnico Scientifica, Pisa, 1961.